Pseudo-Anosovs with small dilatation and the Dehn fillings of the magic manifold

E. Kin* • M. Takasawa
Tokyo Institute of Technology

Workshop on Geometry, Topology and Dynamics of Character Varieties
2010.07.19–30
Pseudo-Anosov (1)

\[ \Sigma = \Sigma_{g,n}; \] orientable surface of genus \( g \) with \( n \) punctures
\[ \text{Mod}(\Sigma); \] the mapping class group of \( \Sigma \).

We focus on pseudo-Anosov elements of \( \text{Mod}(\Sigma) \).
Pseudo-Anosov (2)

- A homeomorphism $\Phi : \Sigma \to \Sigma$ is pseudo-Anosov if
  
  $\exists \lambda = \lambda(\Phi) > 1$ called the dilatation of $\Phi$, and
  
  $\exists \mathcal{F}^s, \mathcal{F}^u$; a pair of transverse measured foliations such that
  
  $\Phi(\mathcal{F}^s) = \frac{1}{\lambda} \mathcal{F}^s$ and $\Phi(\mathcal{F}^u) = \lambda \mathcal{F}^u$.

$\mathcal{F}^s$ and $\mathcal{F}^u$ are called the stable and unstable foliation.
Dilatation, Entropy

The mapping class $\phi = [\Phi] \in \text{Mod}(\Sigma)$ containing a pseudo-Anosov homeo $\Phi$ is called pseudo-Anosov.

- $\lambda(\phi) := \lambda(\Phi) > 1$; dilatation of $\phi$
- $\log \lambda(\phi)$; entropy of $\phi$
- $|\chi(\Sigma)|\lambda(\phi)$; normalized dilatation
- $|\chi(\Sigma)|\log(\lambda(\phi))$; normalized entropy
Minimal dilatation (1)

Fix a surface $\Sigma_{g,n}$.

$$Spec(\Sigma_{g,n}) := \{ \lambda(\phi) \mid \text{pseudo-Anosov } \phi \in \text{Mod}(\Sigma_{g,n}) \}. $$

★ There exists a minimum of $Spec(\Sigma_{g,n})$ (Ivanov)

$$\delta_{g,n} := \min Spec(\Sigma_{g,n})$$

$$\delta_{g} := \delta_{g,0}.$$
Minimal dilatation (2)

**Question 1.** What is the value $\delta_g$ for $g \geq 3$?

★ $\log \delta_g \asymp 1/g$ (Penner, 1991)

**Question 2** (McMullen, 2000).

- Does $\lim_{g \to \infty} g \log \delta_g$ exist?
  
  $\iff$ Does $\lim_{g \to \infty} |\chi(\Sigma_{g,0})| \log \delta_g$ exist?

  minimal normalized entropy of genus $g$

- What is its value?
Magic manifold $N$

- $N := S^3 \setminus (3$ chain link$)$
- hyperbolic, fibered, $\text{vol}(N) = 5.33348\ldots$, the smallest known volume among orientable hyperbolic 3-manifolds with 3 cusps.

We study pseudo-Anosovs which occur as the monodromies on fibers for Dehn fillings of $N$. 
Small dilatation pseudo-Anosovs (1)

For $P > 1$, define

$$
\Psi_P := \{\text{pseudo-Anosov } \Phi : \Sigma \to \Sigma ; \ |\chi(\Sigma)| \log \lambda(\Phi) \leq \log P\}.
$$

Elements of $\Psi_P$ are called the small dilatation pseudo-Anosovs.

For $P$ sufficiently large, (e.g., $P \geq 2 + \sqrt{3}$),

$$
\Psi_P \supset \{\Phi_g : \Sigma_{g,0} \to \Sigma_{g,0}\}_{g \geq 2}
$$

($\Psi_P$ is an infinite set)

**Theorem 1** (Farb-Leininger-Margalit). For any $P > 1$, there exist finite many hyperbolic, fibered 3-mfds $M_1, M_2, \cdots, M_r$ such that any $\Phi \in \Psi_P$ occurs as the monodromy of a Dehn filling of one of the $M_k$. 
Small dilatation pseudo-Anosovs (2)

★ (K-Takasawa, 2009) For each $n = 3, \cdots, 8$ (resp. $n \geq 9$), the pseudo-Anosov homeo. on an $n$-punctured disk with the smallest dilatation (resp. smallest known dilatation) occurs as the monodromy on a fiber for a Dehn filling of $N$.

(Independently, this is also established by Venzke.)
Results

For $r \in \mathbb{Q}$, let $N(r)$ be the Dehn filling of $N$ along the slope $r$.

Main Theorem. (K-Takasawa) Let $r \in \{-\frac{3}{2}, -\frac{1}{2}, 2\}$. Consider the Dehn filling $N(r)$. For any $g \geq 3$, there exists a monodromy $\Phi_g(r) : \Sigma_g \to \Sigma_g$ on a closed fiber of genus $g$ for a Dehn filling of $N(r)$ such that

$$\lim_{g \to \infty} g \log \lambda(\Phi_g(r)) = \log(\frac{3+\sqrt{5}}{2}) = \log(1 + \text{golden ratio}).$$

In particular

$$\lim_{g \to \infty} \sup g \log \delta_g \leq \log(\frac{3+\sqrt{5}}{2}).$$
• Case $r = \frac{-1}{2}$; established by Hironaka

• Case $r = \frac{-3}{2}$; established by Aaber-Dunfield

$N(\frac{-3}{2}) \simeq (-2, 3, 8)$-pretzel link (= Whitehead sister link) exterior
An upper bound of $\delta_g$ given by K-Takasawa and Hironaka

(K-Takasawa) \hspace{1cm} (Hironaka)

$\delta_3 \leq 1.401268, N\left(\frac{-1}{2}, *, *\right)$

$\delta_4 \leq 1.261230, N\left(\frac{-1}{2}, *, *\right)$

$\delta_5 \leq 1.148794, N\left(\frac{-3}{2}, *, *\right)$

$\delta_6 \leq 1.128760, N\left(\frac{-3}{2}, *, *\right)$

$\delta_7 \leq 1.115481, N\left(\frac{-3}{2}, *, *\right)$

$\delta_8 \leq 1.104039, N\left(\frac{-4}{3}, \frac{-25}{17}, \frac{-5}{1}\right)$

$\delta_9 \leq 1.092824, N\left(\frac{-3}{2}, *, *\right)$

$\delta_{10} \leq 1.083766, N\left(\frac{-3}{2}, *, *\right)$

$\delta_{11} \leq 1.077045, N\left(\frac{-3}{2}, *, *\right)$

$\delta_{12} \leq 1.072664, N\left(\frac{-3}{2}, *, *\right)$

$\delta_{13} \leq 1.071692, N\left(\frac{-29}{27}, \frac{-5}{3}, \frac{-6}{1}\right)$

$\delta_{14} \leq 1.062987, N\left(\frac{-3}{2}, *, *\right)$

$\delta_{15} \leq 1.058335, N\left(\frac{-3}{2}, *, *\right)$

$\delta_{16} \leq 1.054998, N\left(\frac{-3}{2}, *, *\right)$

$\delta_{17} \leq 1.052214, N\left(\frac{-3}{2}, *, *\right)$

$\delta_{18} \leq 1.052540, N\left(\frac{-1}{2}, *, *\right)$

$\delta_{19} \leq 1.047084, N\left(\frac{-3}{2}, *, *\right)$
Pseudo-Anosovs with orientable foliations

\[ \delta_g^+ := \min \{ \lambda(\phi) \mid \phi \in \text{Mod}(\Sigma_{g,0}) \text{ with orientable (un)stable foliation} \} \]

Clearly \( \delta_g \leq \delta_g^+ \).

\[ \star \delta_2 = \delta_2^+ \approx 1.722083 \text{ (Zhirov, Cho-Ham)} \]
Minimal dilatation $\delta_g^+$

**Theorem 2** (K-Takasawa, Aaber-Dunfield).

$$\delta_7^+ = \lambda_{(9,2)} \approx 1.11548,$$

where $\lambda_{(k,\ell)}$ is a unique real root greater 1 of the polynomial

$$f_{(k,\ell)}(t) = t^{2k} - t^{k+\ell} - t^k - t^{k-\ell} + 1 \text{ for } k > 0, -k < \ell < k$$

- $\delta_2^+ = \lambda_{(2,1)} \approx 1.72208$ (Zhirov, Cho-Ham)
- $\delta_3^+ = \lambda_{(3,1)} = \lambda_{(4,3)} \approx 1.40127$ (Lanneau-Thiffeault)
- $\delta_4^+ = \lambda_{(4,1)} \approx 1.28064$ (Lanneau-Thiffeault)
- $\delta_5^+ = \lambda_{(6,1)} = \lambda_{(7,4)} \approx 1.17628$ (Lanneau-Thiffeault)
- $\delta_8^+ = \lambda_{(8,1)} \approx 1.12876$ (Hironaka)

**Proposition 1** (K-Takasawa, Aaber-Dunfield).

$$\delta_5 < \delta_5^+. $$
Question on minimal dilatation $\delta_g^+$

Question 3 (Lanneau-Thiffeault). For $g$ even, is $\delta_g^+$ equal to $\lambda_{(g,1)}$?

$\star \delta_2^+ = \lambda_{(2,1)}$ (Zhirov).
$\star \delta_4^+ = \lambda_{(4,1)}$ and $\delta_6^+ \geq \lambda_{(6,1)}$ (Lanneau-Thiffeault).

Proposition 2 (K-Takasawa). If Question 3 is true, then $\delta_g^+ < \delta_{g+1}^+$ for $g \equiv 1, 5, 7, 9 \pmod{10}$ and $g \geq 7$. In particular $\delta_7^+ < \delta_8^+$. 
Background, Idea of Main Theorem

- Thurston norm $X_T : H_2(N, \partial N; \mathbb{R}) \to \mathbb{R}$.

- McMullen’s Teichmüler polynomial $P_\Delta$ on a fiber face $\Delta$.

- Fact: $\exists$ a unique ray on $\text{int}(C_\Delta)$ which realizes

$$\min\{X_T(a) \cdot \log \lambda(a) \mid a \in \text{int}(C_\Delta)\}$$
What we need to show:

- find that minimal ray.
- find a nice sequence of fibers \( \{F_n\} \) so that the ray of homology classes \( \{[F_n]\} \) goes to the minimal ray as \( n \) goes to \( \infty \).
The magic “magic manifold”

★ There are pseudo-Anosov monodromies for Dehn fillings of $N$ realizing

\[ \delta_2, \]
\[ \delta_n^+ \text{ for } n = 2, 3, 4, 5, 7, 8, \]
\[ \delta(D_n) \text{ for } n = 3, 4, \ldots, 8. \]

**Question 4.** Let $g \geq 3$ and $n \geq 9$.

(1) **Is there a pseudo-Anosov monodromy on a closed fiber $\Sigma_g$ realizing $\delta_g$ (resp. $\delta_g^+$) for some Dehn filling of $N$?**

(2) **Is there a pseudo-Anosov monodromy on a fiber $D_n$ realizing $\delta(D_n)$ for some Dehn filling of $N$?**