Local Connectivity of Deformation spaces of Kleinian groups

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Review the definition of $AH(M)$

Topology of the interior

Bumponomics and the failure of local connectivity

A local model for $AH(M)$
Definition of $AH(M)$

Let $M$ be a compact, orientable 3-manifold.

\[ AH(M) = \{ \rho : \pi_1(M) \to PSL(2, \mathbb{C}) \mid \rho \text{ discrete, faithful} \} / PSL(2, \mathbb{C}) \]
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$AH(M)$ inherits a topology as a subset of the character variety

$$X(M) = Hom(\pi_1(M), PSL(2, \mathbb{C}))/PSL(2, \mathbb{C})$$
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\]

If \( P \subset \partial M \) is a collection of annuli and tori

\[
AH(M, P) = \{ \rho \in AH(M) \mid \rho(g) \text{ parabolic for all } g \in P \}
\]
Marked hyperbolic 3-manifolds

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\[ M = S \times I \]
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$\rho \in AH(M) \leadsto N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$

$AH(M)$ is the set of marked hyperbolic 3-manifolds homotopy equivalent to $M$
The interior of $AH(M)$

Theorem (Ahlfors, Bers, Kra, Marden, Maskit, Sullivan, Thurston)

When $\partial M$ is incompressible, the interior of $AH(M)$:
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Each component of the interior is homeomorphic to an open ball
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**Corollary**

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Example:

$$\text{int}(AH(S \times I)) \cong \mathcal{T}(S) \times \mathcal{T}(S')$$
Theorem (Brock, Bromberg, Kim, Kleineidam, Lecuire, Namazi, Ohshika, Souto, Thurston)

\[ AH(M) = \overline{\text{int}(AH(M))} \]

(i.e., every hyperbolic manifold is the algebraic limit of geometrically finite manifolds)
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The interior of \( AH(S \times I) \) self-bumps:
(McMullen, Bromberg, Holt)
The punctured torus

Theorem (Bromberg)

Let $\hat{T}$ denote the punctured torus. Then $AH(\hat{T} \times I, \partial\hat{T} \times I)$ is not locally connected.
The punctured torus

**Theorem (Bromberg)**

Let $\hat{T}$ denote the punctured torus. Then $AH(\hat{T} \times I, \partial \hat{T} \times I)$ is not locally connected.

- Depends on Minsky’s classification of punctured torus groups
AH(S × I) is not locally connected

Theorem (M)

For any closed genus \( g \geq 2 \) surface \( S \), \( AH(S \times I) \) is not locally connected.
For any closed genus $g \geq 2$ surface $S$, $AH(S \times I)$ is not locally connected.

Let $M$ be a hyperbolizable 3-manifold with incompressible boundary containing a primitive essential annulus $A$ and suppose $(\hat{T} \times I, \partial \hat{T} \times I)$ is pared homeomorphic to one of the components $(M', A)$ of $M - A$. Then $AH(M)$ is not locally connected.
Outline of Proof

$$AH(M)$$

\[\text{want to use } \rho \mapsto \rho|_{\pi_1(\hat{T})}\]

$$AH(\hat{T}, \partial\hat{T})$$
Outline of Proof

\[ AH(M) \]

\[ \mathcal{A}_{\hat{T}} \xrightarrow{\Phi} AH(\hat{T}, \partial\hat{T}) \]

- Local model for \( AH(\hat{T}, \partial\hat{T}) \) [Bromberg]
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Local model for (a dense subset of) \( AH(M) \)
Outline of Proof

\[ \mathcal{A}_M \xrightarrow{\Phi} AH(M) \]
\[ \Pi \]
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- Local model for \( AH(\hat{T}, \partial\hat{T}) \) [Bromberg]
- Local model for (a dense subset of) \( AH(M) \)
- Define a map \( \mathcal{A}_M \xrightarrow{\Pi} \mathcal{A}_{\hat{T}} \) by restricting representations \( \mathcal{A}_M \) not locally connected since \( \mathcal{A}_{\hat{T}} \) not locally connected
Outline of Proof

\[ \mathcal{A}_M \xrightarrow{\Phi} AH(M) \]

\[ \Pi \downarrow \]

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- Complex length estimates from Filling Theorem show \( \Phi(\mathcal{A}_M) \) not locally connected. Density implies \( AH(M) \) not locally connected.
Local model for $AH(\hat{T}, \partial \hat{T})$

Given $\sigma$ with extra parabolic

$$\sigma(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$
Local model for $AH(\hat{T}, \partial\hat{T})$

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construct manifold with rank 2 cusp
\[ \sigma_w(c) = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \]
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$A_{\hat{T}} = \{(\sigma, w) \mid \sigma_w \text{ geometrically finite or } w = \infty\}$

$\Phi(\sigma, w) = \begin{cases} 
\text{filling of } \mathbb{H}^3/\sigma_w & \text{if } w \neq \infty \\
\sigma & \text{if } w = \infty 
\end{cases}$
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Theorem (Bromberg)

$\Phi$ extends to a local homeomorphism $\overline{\mathcal{A}_{\hat{T}}} \to AH(\hat{T}, \partial\hat{T})$. 
Hyperbolic Dehn filling

Theorem (Hodgson-Kerckhoff, Bromberg, Brock-Bromberg, M)

Let \( L > 1, \varepsilon > 0 \). There exists \( K \) such that if \(|w| > K\), then

- the hyperbolic Dehn filling of \( \hat{N} \) exists
- the complex length, \( l + i\theta \), of the core curve, \( \gamma \), of the solid filling torus satisfies

\[
\left| l - \frac{4\pi \text{Im}(w)}{|w|^2} \right| \leq \frac{16(2\pi)^3(\text{Im}(w))^2}{|w|^4} \quad \left| \theta - \frac{4\pi \text{Re}(w)}{|w|^2} \right| \leq \frac{10(2\pi)^3(\text{Im}(w))^2}{|w|^4}
\]

- there exists an \( L\)-biLipschitz diffeomorphism

\[
\hat{N} - \{\varepsilon - \text{thin part about cusp}\} \rightarrow N - \{\varepsilon - \text{thin part about } \gamma\}\]
Local model for (most of) $AH(M)$

Given $\sigma$ with two extra parabolics $\sigma(a)$ and $\sigma(b)$
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cusps parametrized by $w_i \in \mathbb{C}$
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Given $\sigma$ with two extra parabolics $\sigma(a)$ and $\sigma(b)$

cusps parametrized by $w_i \in \mathbb{C}$

fill along $c_1$ and $c_2$
Local model for (most of) $AH(M)$

Given $\sigma$ with two extra parabolics $\sigma(a)$ and $\sigma(b)$

\[ \mathcal{A}_M = \{(\sigma, w_1, w_2) \mid \sigma_{w_1,w_2} \text{ geometrically finite or } w_1 = w_2 = \infty \} \]

\[ \Phi(\sigma, w_1, w_2) = \begin{cases} 
\text{filling of } \mathbb{H}^3/\sigma_{w_1,w_2} & \text{if } (w_1, w_2) \neq (\infty, \infty) \\
\sigma & \text{if } (w_1, w_2) = (\infty, \infty) 
\end{cases} \]
Local model for (most of) $AH(M)$

Given $\sigma$ with two extra parabolics $\sigma(a)$ and $\sigma(b)$, extend cusps parametrized by $w_i \in \mathbb{C}$ and fill along $c_1$ and $c_2$.

$$A_M = \{(\sigma, w_1, w_2) \mid \sigma_{w_1, w_2} \text{ geometrically finite or } w_1 = w_2 = \infty\}$$

$$\Phi(\sigma, w_1, w_2) = \begin{cases} \text{filling of } \mathbb{H}^3/\sigma_{w_1, w_2} & \text{if } (w_1, w_2) \neq (\infty, \infty) \\ \sigma & \text{if } (w_1, w_2) = (\infty, \infty) \end{cases}$$

**Theorem (Bromberg)**

*For any $(\sigma, \infty, \infty)$, there is a neighborhood $U$ in $A_M$ such that $\Phi|_U : U \to \Phi(U) \subset AH(M)$ is a homeomorphism.*
Define $\Pi : \mathcal{A}_M \to \mathcal{A}_{\hat{T}}$ by $(\sigma, w_1, w_2) \mapsto (\sigma|_{\pi_1(\hat{T})}, w_1)$.
Define $\Pi : A_M \rightarrow A_{\hat{T}}$ by $(\sigma, w_1, w_2) \mapsto (\sigma|_{\pi_1(\hat{T})}, w_1)$
Define $\Pi : A_M \to A_{\hat{T}}$ by $(\sigma, w_1, w_2) \mapsto (\sigma|_{\pi_1(\hat{T})}, w_1)$.

**Lemma**

There exists a point $(\sigma_0, \infty)$, a neighborhood $U \subset A_{\hat{T}}$, subsets $C_n \subset U$, and some $\delta > 0$ such that for any $(\sigma, w) \in C_n$ and any $(\sigma', w') \in U - C_n$

$$|w - w'| > \delta$$
Define $\Pi : A_M \to A_{\hat{T}}$ by $(\sigma, w_1, w_2) \mapsto (\sigma|_{\pi_1(\hat{T})}, w_1)$

Lemma

There exists a point $(\sigma_0, \infty, \infty) \in U \subset A_M$, subsets $\Pi^{-1}(C_n) \subset U$, and some $\delta > 0$ such that for any $(\sigma, w_1, w_2) \in \Pi^{-1}(C_n)$ and $(\sigma', w_1', w_2') \in U - \Pi^{-1}(C_n)$

$$|w_1 - w_1'| > \delta$$
$AH(M)$ is not locally connected

$\mathcal{A}_M$  

$B_n = \Pi^{-1}(C_n)$  

$\downarrow \Pi$  

$\mathcal{A}_T$  

$C_n$  

$\Phi \quad \rightarrow \quad \Phi(B_n)$
AH(M) is not locally connected

\[ B_n = \Pi^{-1}(C_n) \]

\[ \downarrow \Pi \]

Filling Theorem implies complex length of \( \gamma \) in \( \Phi(\sigma, w_1, w_2) \) is approximately

\[ \ell + i\theta \approx \frac{4\pi \text{Im}(w_1)}{|w_1|^2} + i \frac{4\pi \text{Re}(w_1)}{|w_1|^2} \]
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Filling Theorem implies complex length of $\gamma$ in $\Phi(\sigma, w_1, w_2)$ is approximately

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For all but finitely many $n$, $\Phi(B_n)$ and $\Phi(U - B_n)$ are disjoint
$AH(M)$ is not locally connected

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$\Phi$ implies complex length of $\gamma$ in $\Phi(\sigma, w_1, w_2)$ is approximately

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Density $\Rightarrow$ $AH(M)$ is not locally connected.
Future Directions

- Replace punctured torus with four-punctured sphere
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- At which points is $AH(M)$ locally connected?
Thank you for listening!

Slides and preprints are available at:

www.math.umd.edu/~magid/