Hyperbolic cone-manifold structures with prescribed holonomy

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Geometry, Topology and Dynamics of Character Varieties
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Outline

1. Background
   - Introduction
   - $\text{PSL}_2\mathbb{R}$
   - Euler class of a representation
   - Hyperbolic cone surfaces

2. Statements

3. Ideas in the proofs
   - Punctured tori and pentagons
   - Representation and character varieties
Holonomy

Recall:

- A hyperbolic structure on a manifold $M^n$ is equivalent to an developing map $D : \tilde{M}^n \longrightarrow \mathbb{H}^n$.
- A loop $C \in \pi_1(M, x_0)$ lifts to a path in $\mathbb{H}^n$, giving an isometry $\rho(C)$ relating first and last charts around $x_0$.
- This gives holonomy homomorphism $\rho : \pi_1(M, x_0) \longrightarrow \text{Isom}^+ \mathbb{H}^n$.

Notation:

- Capitals denote curves in $\pi_1(M)$, lower case denotes image under $\rho$, i.e. $\rho(G) = g$.
- All surfaces orientable connected.

2 papers on arxiv

- 1006.5223: Hyperbolic cone-manifold structures with prescribed holonomy I: punctured tori
- 1006.5384: Hyperbolic cone-manifold structures with prescribed holonomy II: higher genus
Holonomy

Recall:
- A hyperbolic structure on a manifold $M^n$ is equivalent to an developing map $\mathcal{D} : \tilde{M}^n \rightarrow \mathbb{H}^n$.
- A loop $C \in \pi_1(M, x_0)$ lifts to a path in $\mathbb{H}^n$, giving an isometry $\rho(C)$ relating first and last charts around $x_0$.
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Questions

\{
\text{Hyperbolic structure on } M
\} \rightarrow \{ \text{Algebraic representation } \pi_1(M) \rightarrow PSL_2\mathbb{R} \}\}

- Which representations \( \pi_1(M) \rightarrow PSL_2\mathbb{R} \) are holonomy maps of hyperbolic structures?
- Do other representations have a geometric interpretation?
- In general, how does algebra determine geometry?
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\end{align*}
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- In general, how does algebra determine geometry?
Known results

- 3-dimensional hyperbolic/euclidean/spherical geometry, $M$ with boundary (but no boundary control): Leleu 2000
- 2-dimensional complex projective geometry, $M$ closed: Gallo–Kapovich–Marden 2000
- 2-dimensional hyperbolic geometry, $M$ closed/punctured/geodesic boundary: Goldman 1980

Here:

2-dimensional hyperbolic geometry.
Extend (and reprove) Goldman’s results.
All the previous results involve lifting representations 
\( \pi_1(M) \rightarrow \text{Isom}^+X \) to the universal cover \( \tilde{\text{Isom}}^+X \).

As unit tangent bundle:

\[
\tilde{PSL}_2\mathbb{R} \cong UTH^2 \cong H^2 \times S^1
\]

\[
\tilde{PSL}_2\mathbb{R} \cong \left\{ \text{“unit tangent bundle but with angles measured in } \mathbb{R} \text{ not } \mathbb{R}/2\pi\mathbb{Z}” \right\} \cong H^2 \times \mathbb{R}
\]

As classes of paths:

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\tilde{PSL}_2\mathbb{R} = \left\{ \text{“Homotopy classes of paths in } PSL_2\mathbb{R} \text{ starting at 1, rel endpoints”} \right\}
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Projection to \( PSL_2\mathbb{R} \): take a path to its endpoint.
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Projection to $PSL_2\mathbb{R}$: take a path to its endpoint.
Lifts to $\tilde{PSL}_2^\mathbb{R}$

Lifts of:
- $1 \in PSL_2^\mathbb{R}$ are $\{z^n : n \in \mathbb{Z}\} = \text{centre} = Z$. $z = 2\pi$ rotation.
- $g \in PSL_2^\mathbb{R}$ are $\tilde{g}Z$ where $\tilde{g}$ is one particular lift.

Lemma

If $g, h \in PSL_2^\mathbb{R}$ then $[g, h]$ is well-defined in $\tilde{PSL}_2^\mathbb{R}$.

A parabolic or hyperbolic $\alpha \in PSL_2^\mathbb{R}$ has a “simplest” lift to $\tilde{PSL}_2^\mathbb{R}$: ‘minimal twist to tangent vector".
Lifts to $\overline{\text{PSL}_2\mathbb{R}}$

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Regions in $\text{PSL}_2\mathbb{R}$

- $\text{Hyp}_0 = \{\text{"simplest lifts of hyperbolics"}\}$
- $\text{Par}_0 = \{\text{"simplest lifts of parabolics"}\} = \text{Par}_0^+ \cup \text{Par}_0^-$
- $\text{Ell}_1 = \{\text{"rotations by } \theta \in (0, 2\pi)\}$
- $\text{Ell}_{-1} = \{\text{"rotations by } \theta \in (-2\pi, 0)\}$
Euler class of a representation

Algebraic definition:

\[ \pi_1(S) = \langle G_1, H_1, \ldots, G_k, H_k \mid [G_1, H_1] \cdots [G_k, H_k] = 1 \rangle \]

Consider \( \rho([G_1, H_1] \cdots [G_k, H_k]) = \begin{cases} 1 \in \text{PSL}_2 \mathbb{R} \\ z^m \in \text{PSL}_2 \mathbb{R} \end{cases} \)

\( m = \text{Euler class of } \rho = e(\rho) \)

Also obstruction-theoretic: “\( e(\rho) \) is the obstruction to an equivariant developing map with vector field \( \mathcal{D} : \tilde{S} \rightarrow UT_{\mathbb{H}^2} \)”

Proposition

\( S \) closed, \( \rho \) holonomy representation. Then \( e(\rho) = \pm \chi(S) \).

For surfaces with boundary, need to trivialize.
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For surfaces with boundary, need to trivialize.
Theorem (Milnor–Wood inequality 1958)

When \( \chi(S) < 0 \), for \( \rho : \pi_1(S) \to \text{PSL}_2\mathbb{R} \)

\[
\chi(S) \leq e(\rho) \leq -\chi(S).
\]

\( e \) is a continuous map from the representation variety to \( \mathbb{Z} \).

\( R(S) = \{ \text{representations } \pi_1(S) \to \text{PSL}_2\mathbb{R} \} \)

Theorem (Goldman 1988)

Suppose \( S \) closed, \( \chi(S) < 0 \). Then \( R(S) \) has \( 2|\chi(S)| + 1 \) components, parametrized by Euler class.

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e = \chi(S), \chi(S) + 1, \ldots, -\chi(S) - 1, -\chi(S).
\]
**Milnor–Wood, Goldman**

**Theorem (Milnor–Wood inequality 1958)**

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**Theorem (Goldman 1988)**

*Suppose* $S$ *closed*, $\chi(S) < 0$. *Then* $R(S)$ *has* $2|\chi(S)| + 1$ *components*, *parametrized by Euler class*.

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Above: for $S$ closed, $\rho$ holonomy representation $\Rightarrow e(\rho) = \pm \chi(S)$ extremal. The converse is also true.

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Consider $\chi(S) < 0$, $\rho : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R}$. If $S$ has boundary, then for each boundary component $C$, assume $\rho(C)$ non-elliptic. TFAE:

1. $\rho$ holonomy of a complete hyperbolic structure on $S$ with geodesic / cusped boundary components (resp. as $\rho$ is hyperbolic or parabolic)
2. $e(\rho) = \pm \chi(S)$

Geometric interpretation for other components? Holonomy of cone-manifold structures.
Geometric interpretation of representations

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Hyperbolic cone surfaces

**Definition**

A surface locally isometric to $\mathbb{H}^2$ except at finitely many singular points. Singular points have neighbourhoods which are:

- a cone on a circle of length $\theta$; interior cone point.
- a cone on an arc of angle $\theta$; boundary cone point or corner point.

*Order* of cone point: excess angle in multiples of $2\pi$.

- of *interior* cone point: $s$ where $\theta = 2\pi(1 + s)$.
- of *boundary* point: $s$ where $\theta = 2\pi\left(\frac{1}{2} + s\right)$. 
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Holonomy of hyperbolic cone surfaces

**Lemma (from Gauss–Bonnet)**

If $S$ is a hyperbolic cone surface, orders of cone points $s_i$, then $\sum s_i < -\chi(S)$.

A loop $C$ around an interior cone point is contractible! So if $\rho$ holonomy, $\rho(C) = 1 \in \text{Isom}^+ \mathbb{H}^2$. But $\rho$ is also rotation by $\theta$. So $\rho : \pi_1(S) \longrightarrow \text{PSL}_2\mathbb{R}$ can be the holonomy of a hyperbolic cone-manifold structure on $S$, but all interior cone angles must be $\in 2\pi\mathbb{N}$.

From obstruction-theoretic definition of Euler class:

**Proposition**

Suppose $\rho$ holonomy of hyperbolic cone-manifold structure on closed $S$, interior cone point orders $s_i$. Then $e(\rho) = \pm (\chi(S) + \sum s_i)$. 
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Statements

When S is a punctured torus...

Theorem (M.)

\( S \) punctured torus, \( \rho : \pi_1(S) \to PSL_2\mathbb{R} \) homomorphism. TFAE:

1. \( \rho \) holonomy for a hyperbolic cone-manifold structure on \( S \) with geodesic boundary except at most one corner point, and no interior cone points;

2. \( \rho \) is not virtually abelian.

Two punctured tori make a closed surface!

Theorem (M.)

\( S \) closed genus 2, \( \rho : \pi_1(S) \to PSL_2\mathbb{R}, e(\rho) = \pm 1 \). Suppose \( \rho \) takes a separating curve to a non-hyperbolic. Then \( \rho \) is the holonomy of a hyperbolic cone surface with one \( 4\pi \) cone point.
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Theorem (M.)

* S closed, genus \( \geq 2 \). Consider representations \( \rho : \pi_1(S) \to PSL_2\mathbb{R} \) with \( e(\rho) = \pm (\chi(S) + 1) \), sending some non-separating simple closed curve to an elliptic. Almost every such representation is the holonomy of a hyperbolic cone-manifold structure on \( S \) with a single cone point, angle \( 4\pi \).

* Almost? There’s a measure on the character variety of representations. Arising from its symplectic structure (Goldman 1984).

* It’s not true that every component of \( R(S) \) contains only cone-manifold holonomy representations.

* Counterexample (Ser Peow Tan 1994): \( S \) closed genus 3, \( e(\rho) = \pm 2 \).
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Results

Theorem (M.)

\( S \text{ closed, genus } \geq 2. \) Consider representations \\
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It’s not true that every component of \( R(S) \) contains only cone-manifold holonomy representations.

Counterexample (Ser Peow Tan 1994): \( S \) closed genus 3, \\
e(\rho) = \pm 2. \)
Let $S$ be a hyperbolic punctured torus, with no interior cone points, one corner point $q$, corner angle $\theta \in (0, 3\pi)$.

Can find two geodesic loops $G, H$, intersecting only at $q$, cutting $S$ into a pentagon; interior angle sum $\theta$. 

\[ G \theta H \] 
\[ p = h^{-1} g^{-1} \overline{q} \]
Punctured tori and pentagons

Pentagon need not be embedded in $\mathbb{H}^2$...

Definition

Given $g, h \in PSL_2\mathbb{R}, p \in \mathbb{H}^2$, the pentagon $P(g, h; p)$ is

$p \rightarrow h^{-1}ghp \rightarrow ghp \rightarrow hp \rightarrow g^{-1}h^{-1}ghp \rightarrow p$.

Lemma (Construction lemma)

$\rho$ is the holonomy of a punctured torus with a corner if and only if $\exists$ a free basis $G, H$ of $\pi_1(S, q)$ and $p \in \mathbb{H}^2$ such that $P(g, h; p)$ is nondegenerate bounding an immersed disc.

To construct punctured tori: just find a good pentagon.
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**Definition**

Given $g, h \in PSL_2\mathbb{R}$, $p \in \mathbb{H}^2$, the pentagon $\mathcal{P}(g, h; p)$ is

$$p \rightarrow h^{-1}ghp \rightarrow ghp \rightarrow hp \rightarrow g^{-1}h^{-1}ghp \rightarrow p.$$ 

**Lemma (Construction lemma)**

$\rho$ is the holonomy of a punctured torus with a corner if and only if $\exists$ a free basis $G, H$ of $\pi_1(S, q)$ and $p \in \mathbb{H}^2$ such that $\mathcal{P}(g, h; p)$ is nondegenerate bounding an immersed disc.

To construct punctured tori: just find a good pentagon.
If $\mathcal{P}(g, h; p)$ works, can vary $p$ and it still works! Obtain many punctured tori with different hyperbolic cone-manifold structures, but same holonomy $\rho$. Cone angle is determined by $g, h, p$ as “twist of commutator”.
Constructing pentagons

Use 2 results.

**Theorem (Nielsen 1918)**

Any automorphism of \( \langle G, H \rangle \) takes \([G, H]\) to a conjugate of itself or its inverse.

**Proposition (Goldman)**

\( Tr[g, h] < 2 \) iff \( g, h \) are both hyperbolic and their axes cross.

- By Nielsen, \( Tr(\rho([G, H])) = t \) is invariant of choice of basis \( G, H \). Go case-by-case on \( t \).
- By Goldman, obtain geometric information from \( g, h \).
Various cases

Case $t < -2$:

- $g, h$ hyperbolic, axes cross, $[g, h]$ hyperbolic also. In fact $\rho$ discrete, complete hyp structure with geodesic $\partial$.

Case $t \in (-2, 2)$:

- $\rho$ holonomy of a (non-punctured!) torus with a cone point. Pentagon degenerate — perturb to nondegenerate.
Various cases

Case $t > 2$:
Need to choose a good basis. Consider action of $MCG(S)$ on character variety.

- Use Markoff triples to get basis with good character.
- Use good character, Goldman & more for explicit construction.
**Character of a \((\text{PSL}_2\mathbb{R})\)-representation \(\rho\):**

\[
X : \pi_1(S) \longrightarrow \mathbb{R}, \quad X(G) = \text{Tr}(\rho(G)).
\]

Trace relations \(\Rightarrow X\) determined by values on a finite subset. **Character variety** \(X(S) = \{\text{characters of all representations}\}\). When \(S\) is a punctured torus:

- \((x, y, z) = (\text{Tr } g, \text{Tr } h, \text{Tr } gh)\) enough: \(X(S) \subset \mathbb{R}^3\).
Punctured torus case

\[ \text{MCG}(S) \cong \text{Out} \pi_1(S) \cong \text{GL}_2 \mathbb{Z} \]

Action of \( \text{GL}_2 \mathbb{Z} \) on \( X(S) \) \( \Rightarrow \) Markoff triples.

\((x, y, z) \sim (x', y', z'):\)

- corresponding representations \( \rho, \rho' \) are conjugate in \( \text{PSL}_2 \mathbb{R} \) after applying an automorphism of \( \pi_1(S) \).

**Proposition**

*For irreducible representations, \((x, y, z) \sim (x', y', z') \iff \text{they can be related by the moves}*

\[(x, y, z) \mapsto \begin{cases} (x, y, xy - z), (-x, -y, z), \text{coordinate permutations} \end{cases} . \]

Dynamics of this \( \text{GL}_2 \mathbb{Z} \)-action are *ergodic* in certain regions (Goldman 2003).

This is the key to structures “almost everywhere”.