On the topology of $\mathcal{H}(2)$

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July 19, 2010
Translation surface

Definition
Translation surface is a flat surface with conical singularities such that the holonomy of any closed curve (which does not pass through the singularities) is a translation of $\mathbb{R}^2$.

Basic properties
- Cone angles at singularities must belong to $2\pi \mathbb{N}$,
- A tangent vector at a regular point can be extended to a parallel vector field,
- Correspondence between a translation surface with a unitary parallel vector field and a holomorphic 1-form on a Riemann surface, zero of order $k \leftrightarrow$ singularity with cone angle $(k + 1)2\pi$. 
Examples

- Flat tori (without singularities)

- Surfaces obtained from polygons by identifying sides which are parallel, and have the same length
\( \mathcal{H}(2) \) is the moduli space of pairs \((M, \omega)\) where \( M \) is a Riemann surface of genus 2 and \( \omega \) is a holomorphic 1-form on \( M \) having only one zero which is of order 2. Equivalently, \( \mathcal{H}(2) \) is the moduli space of translation surfaces of genus 2 having only one singularity with cone angle \( 6\pi \).

**Remark**

- The unique zero of \( \omega \) must be a Weierstrass point of \( M \).
- Every Riemann surface of genus 2 is hyper-elliptic, and therefore has exactly 6 Weierstrass points.
We denote by $\mathcal{M}(2)$ the quotient $\mathcal{H}(2)/\mathbb{C}^*$ which is the set of pairs $(M, W)$, where $M$ is a Riemann surface of genus 2, and $W$ is a marked Weierstrass point of $M$.

A saddle connection on a translation surface is a geodesic segment joining two singularities, which may coincide. In the case of $\mathcal{H}(2)$, every saddle connection is a geodesic joining the unique singularity to itself.
Construction from parallelograms

We represent a parallelogram in $\mathbb{R}^2$ up to translation by a pair of complex numbers $(z_1, z_2)$ such that $\text{Im}(z_1 \bar{z}_2) > 0$. Given three parallelograms $P_1, P_2, P_3$ represented by the pairs $(z_1, z_2), (z_2, z_3)$, and $(z_3, z_4)$ respectively, we can construct a surface in $\mathcal{H}(2)$ by the following gluing:
Construction from parallelograms

Proposition

Every surface in $\mathcal{H}(2)$ can be obtained from the previous construction.

Consequently, on every surface in $\mathcal{H}(2)$, there always exist a family of 6 saddle connections which decompose the surface into 3 parallelograms. We will call such families parallelogram decompositions of the surface.
Construction from parallelograms

Question

- Which triples of parallelograms give the same surface in $\mathcal{H}(2)$?
- Given a surface in $\mathcal{H}(2)$, describe the set of parallelogram decompositions of this surface.
Elementary moves

$T$-move: changing $P_1$
Elementary moves

\textbf{S-move: permuting } (P_1, P_2, P_3) \textbf{.}
Elementary moves

*R*-move: changing $P_2$ and $P_3$
Elementary moves

Theorem

Two triples of parallelograms give rise to the same surface in $\mathcal{H}(2)$ if and only if one can be transformed to the other by a sequence of elementary moves.
Given a surface $\Sigma$ in $\mathcal{H}(2)$, we have elementary moves corresponding to $T$, $S$, $R$ in the set of parallelogram decompositions. Those moves can be realized by homeomorphisms of the surface.

One can associate to each parallelogram decomposition of $\Sigma$ a unique canonical basis of of $H_1(\Sigma, \mathbb{Z})$, then the actions of the corresponding homeomorphisms on $H_1(\Sigma, \mathbb{Z})$ in this basis given by the matrices $T$, $S$, and $R$.

Let $\Gamma$ denote the group generated by $T$, $S$ and $R$. 
The matrices $T, S, R$

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ;$$

$$S = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} ;$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$
Properties of $\Gamma$

- $T$ and $R$ commute, $S^2 = -\text{Id}$,
- $\Gamma \not\subseteq \text{Sp}(4, \mathbb{Z})$, the action of $\Gamma$ on $(\mathbb{Z}/2\mathbb{Z})^4 \setminus \{0\}$ has two orbits, but the action of $\text{Sp}(4, \mathbb{Z})$ is transitive.
- $\Gamma$ is not normal in $\text{Sp}(4, \mathbb{Z})$.
- $\Gamma$ contains $\text{SL}(2, \mathbb{Z})$ as a proper subgroup.
For $g \geq 1$, the **Siegel upper half space** $\mathcal{H}_g$ is the set of $g \times g$ complex symmetric matrices whose imaginary part is positive definite.

The **Jacobian locus** $\mathcal{J}_g$ is the subset of $\mathcal{H}_g$ consisting of period matrices associated to canonical homology bases of Riemann surfaces of genus $g$.

The moduli space $\mathcal{M}_g$ of Riemann surfaces of genus $g$ can be identified with $\mathcal{J}_g/\text{Sp}(2g, \mathbb{Z})$. 
Jacobian locus

- Case $g = 1 : \mathcal{J}_1 = \mathcal{H}_1 = \mathbb{H}$ the hyperbolic upper half plan.
- Case $g = 2 : \mathcal{J}_2 \subsetneq \mathcal{H}_2$, the complement is a countable union of copies of $\mathcal{H}_1 \times \mathcal{H}_1$. 
Main result

**Theorem**

The space $\mathcal{M}(2)$, that is the set of pairs (Riemann surface of genus 2, distinguished Weierstrass point), can be identified with the quotient $\tilde{\mathcal{J}}_2/\Gamma$. 
Main result

Main ideas:

- Generalizing the notion of "parallelogram decomposition" by taking into account the action of the hyperelliptic involution on $\pi_1(M, W)$.
- A connectivity result on a subset of the set of simple closed curves on $M$.
- Hyperellipticity of Riemann surfaces of genus 2, and $\Theta$ function.
Corollary

*We have* $[\text{Sp}(4, \mathbb{Z}) : \Gamma] = 6$.

**Idea:** there exists a map $\rho : \mathcal{M}(2) \rightarrow \mathcal{M}_2$ which is *generically* six to one.
Remark

Let $\text{Mod}_{0,6}$ denote the mapping class group of the sphere with 6 punctures. The fundamental group of $\mathcal{M}(2)$ is the subgroup of $\text{Mod}_{0,6}$ fixing a distinguished puncture.

The universal cover map factors through $\tilde{\mathcal{J}}_2$, therefore we have a surjective homomorphism from $\pi_1(\mathcal{M}(2))$ onto $\Gamma$ (more precisely $\Gamma/\{\pm \text{Id}\}$).