AdS geometry as a tool for Teichmüller theory

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IMS
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Advertise 3d AdS geometry as a tool for Teichmüller theory,

Explain basics of AdS geometry,

State some recent results obtained using AdS,

Examples of proofs,

Some open questions here and there.
Recall that
\[ H^3 = \{ x \in \mathbb{R}^{3,1} \mid \langle x, x \rangle = -1 \& x_0 > 0 \} . \]

\[ AdS_3 = \{ x \in \mathbb{R}^{2,2} \mid \langle x, x \rangle = -1 \} . \]

Lorentz analog of \( H^3 \): complete, constant curvature \(-1\).
From relativity: “Anti de Sitter”, model for gravity (no matter).
Lorentz analog of \( S^3 \): \( PSL(2, \mathbb{R}) \) w/ Killing metric, isometry group, etc.
Basic idea: hyperbolic and AdS 3-mflds as tools for Teichmüller theory.
Hyperbolic 3-manifolds and Teichmüller theory

Based (mostly) on quasifuchsian 3-manifolds. Examples of applications include:

- complex projective structures on surfaces,
- complex earthquakes (McMullen),
- the volume of the convex core of quasifuchsian manifolds is coarsely equivalent to the Weil-Petersson distance between the metrics on its boundary (Brock),
- the renormalized volume as a Kähler potential for WP,
- properties of the grafting map.

Not developed here.
AdS 3-manifolds and Teichmüller theory

Some aspects:
- earthquakes,
- extensions of the earthquake flow,
- minimal Lagrangian diffeos.

AdS side involves physically relevant notions:
- globally hyperbolic (GH) spaces (analogs of quasifuchsian),
- “particles”,
- multi-black holes,
- maximal surfaces.

Notations: $S$ closed surface of genus $\geq 2$, $\mathcal{T}$ Teichmüller space.
Measured laminations

$\mathcal{WM} = \{ \text{weighted multicurves on } S \}$ : set of disjoint simple closed curves, each with a positive weight.

$\mathcal{WM}$ is infinite : simple closed curves on $S$ can wrap around a lot.

Let $(c_i, l_i)_{i=1,\ldots,n} \in \mathcal{WM}$, the $c_i$ form a lamination and the $l_i$ define a transverse measure : gives a total weight to $\gamma$, transverse to the $c_i$.

This gives a topology to $\mathcal{WM}$.

The completion of $\mathcal{WM}$ is the space of measured laminations $\mathcal{ML}$.

Measured laminations can be pretty complicated.

\begin{itemize}
  \item $\mathcal{ML} \simeq \mathbb{R}^{6g-6}$.
  \item $\partial T \simeq \mathcal{ML}/\mathbb{R}_{>0}$ (Thurston).
  \item $T \times \mathcal{ML} \simeq T^*T$.
\end{itemize}
Thurston’s Earthquake Thm

Start with a hyperbolic surface. If \( w \in \mathcal{ML} \) is a weighted curve and \( h \in \mathcal{T} \), \( E_l(w)(h) \) is obtained by realizing \( w \) as a geodesic in \( h \), cutting \( S \) open along \( w \), turning the left-hand side by the weight, and gluing back. Defines a homeomorphism

\[
E_l(w) : \mathcal{T} \to \mathcal{T}.
\]

Extends by continuity to \( E_l : \mathcal{T} \times \mathcal{ML} \to \mathcal{T} \) (Thurston).

**Thm** (Thurston, Kerckhoff). \( \forall h, h' \in \mathcal{T}, \exists! \lambda \in \mathcal{ML}, h' = E_l(\lambda)(h) \).

Simple proof by Mess (1990) based on AdS geometry.
Extensions of the Earthquake Thm

Extension of the Earthquake Theorem:

- to hyperbolic surfaces with cone singulars of angle $< \pi$. (w/ Francesco Bonsante.)

- to hyperbolic surfaces with geodesic boundary: $2^N$ earthquakes sending $h$ to $h'$. (w/ Bonsante, Kirill Krasnov).

The proof of the 1st statement is equivalent to an extension of the Mess parameterization for GH AdS manifolds with “particles” : cone singularities along time-like lines, $\theta < \pi$. The analogous quasifuchsian statement holds : Bers-type theorem for quasifuchsian manifolds with cone singularities of $\theta < \pi$ along infinite lines (LeCueur, Moroianu, S.).

The 2nd statement is based on multi-black holes : like globally hyperbolic manifolds, based on a complete, non-compact surface. AdS analogs of Schottky mflds.
**Dynamics of earthquakes**

**Thm** (Bonsante, S.). Let $\lambda, \mu \in \mathcal{ML}$ that fill $S$. Then $E_r(\lambda) \circ E_r(\mu)$ has a fixed point on $T$.

Uniqueness?
See talk by Francesco.
A cyclic extension of the earthquake flow

For $\lambda \in \mathcal{ML}$ fixed, $E_l(\lambda)$ defines an action of $\mathbb{R}$ on $\mathcal{T}$, by $(t, h) \mapsto E_l(t\lambda)(h)$. Analog of horocyclic flow.

We define (w/ Bonsante & Gabriele Mondello) an “extension” : equivalently

- for $c \in \mathcal{T}$, $C_c : S^1 \times \mathcal{T} \to \mathcal{T}$,
- an action $D$ of $S^1$ on $\mathcal{T} \times \mathcal{T}$.

3 (related) definitions based on

- GH AdS 3-mflds,
- minimal Lagrangian maps,
- holomorphic quadratic differentials.
Properties of the cyclic flow

Some properties:

- Limits to the earthquake flow: if $t_n h^*_n \to \lambda$ then $D_{t_n}(h, h^*_n) \to E_l(\lambda/2)(h)$.

- Extension of the earthquake thm:
  \[ \forall \theta \in S^1 \setminus \{0\}, \forall h, h' \in T, \exists! c \in T, C_c(\theta, h) = h'. \]

- Has a complex extension, which limits to McMullen’s complex earthquakes.

- Extends to a $S^1$ action on the universal Teichmüller space.

The extension of the Earthquake Thm follows from a recent result of Barbot, Béguin and Zeghib on constant Gauss curvature foliations of AdS manifolds.
A homeo of $S^1$ is \emph{quasi-symmetric} if it is the boundary of a quasi-conformal diffeo of the disk.

**Def.** $\mathcal{T}_U =$ space of quasi-symmetric orientation-preserving homeos of $S^1$, up to $\text{PSL}(2, \mathbb{R})$.

Let $\rho_0 \in \mathcal{T}$, then any $\rho \in \mathcal{T}$ is conjugated to $\rho_0$ by a quasi-conformal diffeo $\phi$. Moreover $\partial \phi$ is unique. Therefore all $\mathcal{T}$ embed in $\mathcal{T}_U$.

**Question:** canonical quasi-conformal extension(s) to the disk of a quasi-symmetric homeo?

**Conj** (Schoen). Any quasi-symmetric homeo of $S^1$ has a unique quasi-conformal harmonic extension to the disk.

Uniqueness. Partial results on existence. True for closed surfaces.
**Def.** A diffeomorphism \( \phi : H^2 \to H^2 \) is *minimal Lagrangian* if it is area-preserving and its graph is minimal in \( H^2 \times H^2 \).

\( \phi \) is minimal Lagrangian iff \( \phi = v \circ u^{-1} \), where \( u, v : D \to H^2 \) are harmonic maps with opposite Hopf differentials. “Squares” of harmonic map.

**Thm** (Bonsante, S). Any quasi-symmetric homeomorphism \( h \) of \( S^1 \) has a unique extension as a quasi-conformal minimal Lagrangian diffeomorphism of \( H^2 \).

Known (Schoen, Labourie 1992) for closed surfaces. Also when \( h \) has small dilation (Aiyama, Akutagawa, Wan 2000).
AdS$_3$ as a Lorentz analog of $H^3$

\[ \text{AdS}_3 = \{ x \in \mathbb{R}^{2,2} \mid \langle x, x \rangle = -1 \} . \]

Constant curvature $-1$, $\pi_1(\text{AdS}_3) = \mathbb{Z}$.

- Conformal model, in a cylinder.
- Projective model, in a quadric.
- Space-like, time-like, light-like directions. Time-like geodesics are closed of length $2\pi$.
- Totally geodesic space-like planes $\simeq H^2$.
- $\text{Isom}(\text{AdS}_3) = O(2,2)$.
- Boundary at $\infty$ with Lorentz-conformal structure.
AdS$_3$ as a Lorentz analog of $S^3$

Recall: $S^3 = SU(2) \simeq SO(3)$, and $Isom(S^3) = O(4) \simeq O(3) \times O(3)$. $AdS_3 = PSL(2, \mathbb{R})$ with its Killing metric. Left and right actions of $PSL(2, \mathbb{R})$, identifies $Isom_0(AdS_3) = PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ (up to index 2).

Geometrically:

- $\partial_\infty AdS_3$ is foliated by 2 families of lines.
- Thus $\partial_\infty AdS_3 \simeq \mathbb{R}P^1 \times \mathbb{R}P^1$.
- Isometries act projectively on each family,
- Space-like curves in $\partial_\infty AdS_3$ are graphs of functions $\mathbb{R}P^1 \to \mathbb{R}P^1$. 
Globally hyperbolic AdS manifolds

**Def.** An AdS mfd $M$ is *maximal globally hyperbolic* if

- it contains a closed, space-like surface $S$,
- any inextendible time-like curve intersects $S$ exactly once,
- it is maximal (for inclusion) under those properties.

Then $M \simeq S \times \mathbb{R}$, and $M = \Omega / \rho(\pi_1 S)$, where $\Omega \subset AdS_3$. GH AdS mfd's are strongly reminiscent of quasifuchsian hyperbolic mfd's, but in a way more relevant to Teichmüller theory (Mess).
$M$ has a “limit set” $\Lambda_\Gamma$, which is a Jordan curve. $\Lambda_\Gamma = \partial \Omega \cap \partial_\infty \text{AdS}_3$.
$M$ has a “convex core”, $C(M) = CH(\Lambda_\Gamma)/\Gamma$.
It has two boundary components, both with hyperbolic induced metrics $m_\pm$, bent along measured laminations $l_\pm$ that fill (Mess).

**Question** (Mess). can any $m_\pm$ be uniquely realized?
Existence seems to hold (Boubacar Diallo, in progress). **Uniqueness**?

**Thm** (Bonsante, S.) Any $l_-$, $l_+$ that fill can be realized. **Uniqueness**?
A Bers-type parametrization

Given a GHMC AdS mfld $M$, $\rho : \Gamma \to SO(2, 2) \simeq PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$. So, $(\rho_L, \rho_R) : \Gamma \to PSL(2, \mathbb{R})$.

**Thm** (Mess).

- $\rho_L, \rho_R$ have maximal Euler number.
- The map $GH \to T \times T$ is a homeomorphism.

The hyperbolic metrics $c_L, c_R$ corresponding to $\rho_L, \rho_R$ are analogs of the conformal metrics at infinity.
Proof of the Earthquake Thm

$m_{\pm}$ are related to $c_l, c_r$ by earthquakes along $l_{\pm}$. The Earthquake thm follows from this by simple arguments.

- Fix $c_l, c_r$. By Mess’ thm, there are unique $m_{\pm}, l_{\pm}$.
- $c_r = E_r(l_+) \circ E_l(l_+)^{-1}(c_l)$
- $E_l(l_+)^{-1} = E_r(l_+)$,
- so $E_r(l_+) \circ E_l(l_+)^{-1} = E_r(2l_+)$.
- Thus $c_r = E_r(2l_+)(c_l)$, and similarly $c_r = E_l(2l_+)(c_l)$.
- Uniqueness follows from the same argument.

The existence of fixed points of $E_l(\lambda) \circ E_l(\mu)$ follow similarly from prescribing $l_-, l_+$. 
Maximal surfaces in AdS

Let $\Sigma \subset AdS_3$ be a space-like graph. We call:

- $I$ the induced metric, $J$ its complex structure,
- $B$ the shape operator, $BX = -\nabla_X N$,
- $E$ the identity.

**Def.** $h_L, h_R = I((E \pm JB)\cdot, (E \pm JB)\cdot)$.

**Prop** (Krasnov, S.). if $\Sigma$ has principal curvatures $|k_i| < 1$ then $h_L, h_R$ are hyperbolic metrics. If $h_L, h_R$ are complete, we obtain $\phi : H^2 \to H^2$.

Related to the left/right representations for GH mfdls.

**Prop.** $\Sigma$ is *maximal* iff $\phi$ is min Lagrangian. It is quasi-conformal iff $|k_i| < 1$ uniformly.

**Prop.** If in addition $\partial_\infty \Sigma$ is the graph of a quasi-symmetric homeo $\subset \partial_\infty AdS_3 \simeq \mathbb{RP}^1 \times \mathbb{RP}^1$, then $h_L, h_R$ are complete and $\partial_\infty \Sigma$ is the graph of $\phi$. 

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**Statement on maximal surfaces**

**Thm B** (Bonsante, S). Let $\Gamma \subset \partial_{\infty} AdS_3$ be the graph of a quasi-symmetric homeo. Then there exists a unique maximal surface $\Sigma \subset AdS_3$ with $|k_i| < 1$ uniformly such that $\partial_{\infty} \Sigma = \Gamma$.

Thm A follows through the correspondance with min Lagrangian maps. Thm B has a partial extension to higher dimensions (existence).

The key step in the proof of Thm B are compactness estimates for maximal surfaces in $AdS_n$, using results of Barnik (1984).
AdS geometry and its applications to Teichmüller theory remains relatively open.

- Open questions on the boundary of the convex core of GH mflds, and applications to earthquakes.
- Use AdS to prove Schoen’s conjecture on harmonic extensions?
- Extend to AdS setting various results known for quasifuchsian mflds?
- Other questions and applications, not yet discovered??
Thanks for your attention!