On Discreteness of Commensurators

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Summary and Motivation

**Theorem**

*(Margulis) An irreducible lattice \( \Gamma \) in a semi-simple Lie group \( L \) is arithmetic iff the commensurator \( \text{Comm}(\Gamma) \) is dense.*

\[
\text{Comm}(\Gamma) = \{ g \in L : g\Gamma g^{-1} \cap \Gamma \text{ is of finite index in both } \Gamma, g\Gamma g^{-1} \}\]

**Question:** (Shalom) If \( \Gamma \) is a Zariski dense, infinite covolume, discrete subgroup of a semi-simple Lie group \( L \), describe \( \text{Comm}(\Gamma) \). (i.e. Is it discrete?)
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**Question:** (Shalom) If $\Gamma$ is a Zariski dense, infinite covolume, discrete subgroup of a semi-simple Lie group $L$, describe $\text{Comm}(\Gamma)$. (i.e. Is it discrete?)
Answer: (M–) Yes, if

a) The limit set $\Lambda_\Gamma \subset \partial F G$ (=Furstenberg boundary) is not invariant under a simple factor, OR

b) $\Gamma$ is finitely generated and $G = \text{PSL}_2(\mathbb{C})$.

Theorem

(Greenberg '74) If $\Gamma$ is a Zariski dense, finitely generated, infinite covolume, discrete subgroup of $G = \text{PSL}_2(\mathbb{C})$, and $\Lambda_\Gamma \neq S_\infty^2$ then $\text{Comm}(\Gamma)$ is discrete.
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(Leininger, Long, Reid, ’09) If $\Gamma$ is a Zariski dense, finitely generated, infinite covolume, discrete subgroup of $G = \text{PSL}_2(\mathbb{C})$, such that $\Gamma$ is non-free and without parabolics, then $\text{Comm}(\Gamma)$ is discrete.

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Non-full Limit sets

- $\Gamma$ Zariski-dense infinite covolume subgroup of a semi-simple Lie group $L = Isom(X)$.
- $X$ a rank one symmetric space.
- Let $\text{Comm}(\Gamma)$ be the closure of $\text{Comm}(\Gamma)$.
- $L_0 = \text{connected component of the identity, with Lie algebra } l_0$–invariant under adjoint representation.
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(CRUCIAL!) If $L_0$ is non-compact, then
\[ \Lambda_{L_0} = \Lambda_{\text{Comm}(\Gamma)} = \Lambda_{\Gamma} \] is invariant under $L_0$.

Zariski density implies $L_0 = L$. Hence $\Lambda_{\Gamma} = \partial X$.

$L_0$ compact. $L_0$ fixes some point $x \in X$. $L_0$ is normal in $L$. Therefore $L_0$ fixes all $x \in X$. Therefore $L_0$ is trivial.
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Non-full Limit sets–Higher Rank

Theorem

(Benoist ’97) Let $\Gamma \subset G = \text{Isom}(X)$ be a Zariski dense subgroup. Then $\Lambda_\Gamma$ is the unique minimal closed $\Gamma$-invariant subset of the Furstenberg boundary $G/P$.

This Theorem allows us to push through the crucial step in the previous page.

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Full Limit Sets–Kleinian Groups  
For the rest of the talk, $\Gamma$-f.g. Kleinian group with $\Lambda_\Gamma = S^2_\infty$. Then $G$ (as an abstract group) is hyperbolic relative to its parabolic subgroups.  
For concreteness: $G =$ surface group with or without parabolics.
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**Theorem** (M–) $G$ – f.g. Kleinian group. $i : \Gamma_G \rightarrow \mathbb{H}^3$ identifies Cayley graph of $G$ with orbit of a point in $\mathbb{H}^3$.

Then $i$ extends continuously to a map $\hat{i} : \hat{\Gamma}_G \rightarrow \mathbb{D}^3$, where $\hat{\Gamma}_G$ denotes the (relative) hyperbolic compactification of $\Gamma_G$.

Let $\partial i$ denote the restriction of $\hat{i}$ to the boundary $\partial \Gamma$ of $\Gamma$.

Then $\partial i(a) = \partial i(b)$ for $a \neq b \in \partial \Gamma$ if and only if $a, b$ are either ideal end-points of a leaf of an ending lamination of $G$, or ideal boundary points of a complementary ideal polygon.
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Cannon-Thurston Relations

A **Cannon-Thurston map** \( \hat{i} \) from \( \hat{G} \) to \( \hat{X} \) is a continuous extension of \( i \). The restriction of \( \hat{i} \) to \( \partial G \) will be denoted by \( \partial i \). The map \( \partial i \) induces a relation \( R_{CT} \) on \( \partial G \) where \( x \sim y \) if \( \partial i(x) = \partial i(y) \) for \( x, y \in \partial G \).

*Distinct* pairs of points identified by \( \partial i \) will be denoted as \( R_{2CT} \), which is a subset of \( \partial^2(G) \).

\( R_{CT} \) is a closed relation on \( \partial G \).

**Lemma**

Suppose \( G \) acts on \( X \) without accidental parabolics. If \( (x, y) \in R_{CT} \) and \( x \neq y \), then \( x \) cannot be a pole of \( G \).
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Mahan Mj
Cannon-Thurston Relations (Contd.)
Density of Orbits of cosets of $\mathcal{R}_{CT}$ in the Hausdorff metric:
Let $K \subset \mathcal{R}_{CT}$ be a coset (equivalence class) of the relation.
Let $C_c(\partial G)$ denote the space of closed subsets of $\partial G$ with the Hausdorff metric.
Then for all $x \in \partial G$, the singleton set $\{x\}$ is an accumulation point of $\{g.K : g \in G\}$. 

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\( \bar{f} \in \text{Comm}(G) \) implies \( f \in \text{Homeo}(\partial G) \).

"Non-proof": Pull \( \text{Comm}(G) \)-action back to \( \partial G \). Then \( \text{Comm}(G) \) preserves closed totally disconnected relation. Hence its a closed totally disconnected subgroup of \( L \)--discrete.

Let \( f_n \) be a sequence of homeomorphisms of \((\partial G, d)\) that preserves the cosets of \( R_{CT} \), where \( d \) denotes some visual metric.

Let \( \bar{f}_n \) denote the induced homeomorphisms of \( \Lambda_G \).

If \( f_n \to id \) in the uniform topology on \( \text{Homeo}(\partial G) \) then \( \bar{f}_n \to id \) in the uniform topology on \( \text{Homeo}(\Lambda_G) \).

Conversely, if \( \bar{f}_n \to id \) in the uniform topology on \( \text{Homeo}(\Lambda_G) \) then for every pole \( p \in \partial G \), \( d(p, f_n(p)) \to 0 \).
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 Totally Degenerate Surface Groups

**Lemma**

Let $\overline{f}_n \in \text{Comm}(H)$ be a sequence of commensurators converging to the identity in $\text{Isom}(\mathbb{H}^3)$ and let $f_n$ be the induced homeomorphisms on the (relative) hyperbolic boundary $\partial \pi_1(S)(= S^1)$ of the group $\pi_1(S)$. Then $f_n \to \text{Id} \in \text{Homeo}(S^1)$.

Proof Idea Follows.
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Theorem

Let $H$ be a totally degenerate surface Kleinian group. Then the commensurator $\text{Comm}(H)$ of $H$ is discrete in $\text{PSL}_2(\mathbb{C})$.

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Totally Degenerate Surface Groups
By previous Lemma, for any ideal polygon $\Delta$ with boundary in the ending lamination there exists $N = N(\Delta)$ such that $f_n$ fixes all the vertices of $\Delta$ for all $n \geq N$.

Let $z_\Delta \in S^2_\infty$ be the common image of the end-points of $\Delta$ under the Cannon-Thurston map.

Choose ideal polygons $\Delta_1, \cdots, \Delta_k$ such that the common images $\{z_1, \cdots, z_k\}$ is Zariski dense in $S^2_\infty$.

Hence for all $n \geq \max_{i=1 \cdots k}\{N(\Delta_i)\}$, $f_n = Id$. □
Totally Degenerate Surface Groups

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Let $z_\Delta \in S^2_\infty$ be the common image of the end-points of $\Delta$ under the Cannon-Thurston map.

Choose ideal polygons $\Delta_1, \cdots, \Delta_k$ such that the common images $\{z_1, \cdots, z_k\}$ is Zariski dense in $S^2_\infty$.

Hence for all $n \geq \max_{i=1 \cdots k}\{N(\Delta_i)\}$, $f_n = Id$. □
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Hence for all $n \geq \max_{i=1}^k \{N(\Delta_i)\}$, $\overline{f_n} = Id$. □