Supersymmetric Surface Operators, Four-Manifold Theory, and Invariants in Various Dimensions

arXiv:1006.3313

Meng-Chwan Tan
NUS
Organization of Talk

• A summary of established mathematical results that we shall prove purely physically.

• A summary of the new mathematical results as suggested by the underlying physics.

• Surface operators and singular Donaldson invariants in mathematics and physics.
Organization of Talk

• Explain the physical proofs of the established mathematical results mentioned in pt. 1.

• Explain the underlying physics which leads to the new mathematical results mentioned in pt. 2

• Conclusion
Established Results To Be Proved Physically

- Adjunction inequality for embedded surfaces of negative self-intersection in 4-manifolds (Oszvath-Szabo).

- A relation among the Seiberg-Witten invariants of certain 4-manifolds. (Oszvath-Szabo).

- $SW = Gr$ (Taubes).
New Mathematical Results From the Underlying Physics

• Identities among the Gromov-Taubes invariants.

• Affirming a knot-homology conjecture by Kronheimer-Mrokwaw.
New Mathematical Results From the Underlying Physics

• The Gromov-Taubes invariants on symplectic 4-manifold $X = M \times S^1$, and the instanton Floer homology and the Casson-Walker-Lescop invariant of $M$.

• The monopole Floer homology and Seiberg-Witten invariants of $M$. 
Surface Operators

• Introduces a singularity in the SU(2) gauge field along a two-surface \( D \) in the 4-manifold.

\[ A = \alpha d\theta + \cdots \]

• Nontrivial holonomy \( \exp(2\pi \alpha) \) along a small circle linking \( D \).

• Decomposition of gauge bundle along \( D \)

\[ E = L \oplus L^{-1} \]
Surface Operators

- Dimension of moduli space of anti-self-dual connections:
  \[ s = 8k - \frac{3}{2}(\chi + \sigma) + 4l - 2(g - 1) \]

- Instanton number:
  \[ k = -\frac{1}{8\pi^2} \int_X \text{Tr} F \wedge F \]

- Monopole number:
  \[ l = -\int_D c_1(L) \]
Singular Donaldson Invariants

\[ \mathcal{D}'_E : H_0(X \setminus D, \mathbb{R}) \oplus H_2(X \setminus D, \mathbb{R}) \to \mathbb{R}. \]

\[ \mathcal{D}'_E(p, S) = \sum_{2m+4t=s} S^m p^t d_{m,t}^{k'}, \]

\( p \in H_0(X \setminus D, \mathbb{R}) \to \Omega^0(p) \in H^4(\mathcal{M}'), \)

\( S \in H_2(X \setminus D, \mathbb{R}) \to \Omega^2(S) \in H^2(\mathcal{M}'), \)

\[ d_{m,t}^{k'} = \int_{\mathcal{M}'} [\Omega^0(p)]^t \wedge \Omega^2(S_{i_1}) \wedge \cdots \wedge \Omega^2(S_{i_m}). \]
Singular Donaldson Invariants

- Physical interpretation via N=2 topological quantum field theory with SU(2) gauge symmetry.

\[ \langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \int D\Phi \mathcal{O}_1 \ldots \mathcal{O}_n e^{-S_E}, \]

\[ S_E = \left\{ \frac{Q, V}{e^2} \right\} + \frac{i \Theta}{8\pi^2} \int_X \text{Tr} F' \wedge F' - i \int_X \text{Tr} \eta \delta_D \wedge F' \]
Singular Donaldson Invariants

- \( I'_0(p) = \frac{1}{8\pi^2} \text{Tr} (\phi(p))^2 \)

- \( I'_2(S) = -\frac{1}{\sqrt{32\pi^2}} \int_S \text{Tr} ((\phi)F)'_0 \)

- \( \langle [I'_0(p)]^t I'_2(S_{i_1}) \cdots I'_2(S_{i_m}) \rangle_{k'} = \int_{\mathcal{M}'_{k'}} [I'_0(p)]^t \wedge I'_2(S_{i_1}) \wedge \cdots \wedge I'_2(S_{i_m}) \)

- \( Z'_{\xi,g}(p, S) = \sum_{k'} \sum_{m \geq 0, t \geq 0} \frac{S^m}{m!} \frac{p^t}{t!} d^{k'}_{m, t} = \sum_{k'} \langle e^{p I'_0 + I'_2(S)} \rangle_{k'} \)

- \( D'_{\xi}(S) = \sum_{k'} \left( 1 + \frac{1}{2} \frac{\partial}{\partial p} \right) \cdot \langle e^{p I'_0 + I'_2(S)} \rangle_{k'} \bigg|_{p=0} \)
Physics Proofs of Established Mathematical Results

• Adjunction inequality for embedded surfaces of negative self-intersection in 4-manifolds (Oszvath-Szabo).

1. \[ 2D \cap D - (2g - 2) \leq 4l \leq (2g - 2). \]

2. \[ g \geq 1 \quad l = \int_D F_L/2\pi. \]

3. \[ (2g - 2) \geq D \cap D + c_1(L_d^2)[D]. \]
Physics Proofs of Established Mathematical Results

4. $\Sigma \cap \Sigma \leq 0 \quad (2g - 2) \geq \Sigma \cap \Sigma - c_1(L_d^2)[\Sigma],$

5. $|c_1(L_d^2)[\Sigma]| + \Sigma \cap \Sigma \leq (2g - 2).$

• Easier formula for surfaces of positive self-intersection proved by KM.

• Generalization of Thom’s conjecture for $\mathbb{CP}^2$. 
Physics Proofs of Established Mathematical Results

• A relation among the SW invariants. (Oszvath-Szabo)

1. $X$ is a compact, oriented, symplectic four-manifold with $b_1 = 0$ and $b_2^+ > 1$.

2. $Z'_D = 2^{1 + \frac{2\chi}{4} + \frac{11p}{4}} \left\{ \sum_{\lambda} SW_{\lambda} e^{2p + S^2/2} e^{2(S+\Sigma,\lambda)} + i(\chi + \sigma)/4 \sum_{\lambda'} SW_{\lambda'} (-1)^{a^2\Sigma^2} e^{-2p - S^2/2 - \Sigma^2} e^{-2i(S+i\Sigma,\lambda)} \right\}.$
Physics Proofs of Established Mathematical Results

3. \[ \alpha = \pm 1, \]
\[ \sum_{\lambda} \left\{ SW(\lambda) e^{2p+S^2/2} e^{2(S,\lambda)} + i^{(\chi+\sigma)/4} SW(\lambda) e^{-2p-S^2/2} e^{-2i(S,\lambda)} \right\} \]
\[ = \sum_{\lambda} \left\{ SW(\lambda) e^{2p+S^2/2} e^{2(S,\lambda)} + i^{(\chi+\sigma)/4} SW_{\lambda'} e^{-2p-S^2/2} e^{-2i(S,\lambda)} \right\}. \]

4. \[ SW_{s'} = SW(s) \]
\[ s' = -i\lambda' \text{ and } s = -i\lambda; \quad \lambda' = \lambda + i\delta \Sigma; \]
\[ SW_{\lambda'} = \int_{M_{sw}^{\lambda'}} (a_d)^{dL_d^2/2}, \quad SW(\lambda) = \sum_{x_i} (-1)^{n_i}, \]
Physics Proofs of Established Mathematical Results

• \( Gr = SW \ (Taubes) \)

1. \( X \) is a compact, oriented, symplectic four-manifold with \( b_1 = 0 \) and \( b_2^+ > 1 \).

2. \[
\sum_{p'} \mathcal{D}^p_0 (S) = \sum_{\lambda} SW(\lambda) e^{2(S+\Sigma,\lambda)+S^2/2+f(x+\sigma)},
\]

3. \[
\mathcal{D}^k_0 (S) = \mathcal{D}^k_0 (0) = \langle 1 \rangle_{k'} .
\]

4. \[
SW(s) = \langle 1 \rangle_{k'}
\]
Physics Proofs of Established Mathematical Results

5. Let us now send the effective value of $\alpha$ to $+1$;

6. $\langle 1 \rangle_{k'} = \sum_x \text{sign} (\det D)$, $c_1 (\mathcal{E}) = \delta_\Sigma$.

7. $\text{SW} (\hat{s}) = \pm \text{Gr} (c_1 (\mathcal{E}))$,

   $\hat{s} = \frac{1}{2} c_1 (\mathcal{L})$,

   $\mathcal{L} = K^{-1} \otimes \mathcal{E}^2$. 
New Math Results From Physics

• Relations among the Gromov-Taubes invariants.

1. $X$ is a compact, oriented, symplectic four-manifold with $b_1 = 0$ and $b_2^+ > 1$.

2. $SW(s) = \text{Gr}(c_1(\mathcal{E}))$, $s = \frac{1}{2}c_1(K^{-1} \otimes K^2)$

3. Via pt. 7 in last slide and pt. 2 above

$$\text{Gr}(c_1(K)) = \pm \text{Gr}(c_1(\mathcal{E})).$$
New Math Results From Physics

4. \( \text{Gr}(0) = 1 \) by definition, so from pt. 7 in 2\textsuperscript{nd} last slide, we have \( \text{SW}(s) = 1 \), and so from pt. 2 above, we have

\[
\text{Gr}(c_1(\mathcal{E})) = +1.
\]

5. Since \( \text{SW}(\overline{s}) = \pm \text{SW}(-\overline{s}) \), from pt. 7 in 2\textsuperscript{nd} last slide, and pt. 3 above, we have

\[
\text{Gr}(c_1(K)) = \pm \text{Gr}(c_1(K) - c_1(\mathcal{E})).
\]
New Math Results From Physics

• Affirming a knot homology conjecture by KM.

1. General $X = M \times S^1$, where $M = \text{compact, closed}$ 3-manifold and $b_1(M) > 1$.

2. Amenable to supersymmetric quantum mechanical interpretation. Ground states contribute to partition function.

3. Consider surface operator $D = S^1 \times K$
New Math Results From Physics

4. \[ \langle 1 \rangle_{k'} = \chi(HF_*(M; K; \alpha)) \]

5. \[ \chi(HF_*(M; K; \alpha)) = \sum_x \text{sign}(h''(x)) = \sum_x \pm 1, \]

\[ \rho : \pi_1(M\setminus\tilde{K}) \to SU(2) \]

6. Can identify \( HF_*(M; \tilde{K}; \alpha) \) with \( LI_*(M, \tilde{K}) \) defined by KM
New Math Results From Physics

7. Since non-trivial phase will be picked up by gauge field at knot crossings, phase is trivial along the longitude of unknot $K_0$.

8. This implies that $LI_{*}(M, K)$ is zero only for unknot.

9. So, $\chi(LI_{*}(M, K))$ vanishes only when the symmetrized Alexander polynomial is trivial.
New Math Results From Physics

• The Gromov-Taubes invariants, instanton Floer homology, and the Casson-Walker-Lescop invariant.

1. Symplectic $X = M \times S^1$, where $M$ = compact, closed 3-manifold and $b_1(M) > 1$.

2. $\alpha = +1$, via $\langle 1 \rangle_{K'} = \chi(HF_*(M; K; \alpha))$ (pt.4 in 2nd last slide) we have

$$\text{Gr}(c_1(\mathcal{E})) = \chi(HF_*(M)).$$
New Math Results From Physics

3. Agree with known mathematical results for

\[ X = \Sigma_g \times T^2 \text{ where } g \geq 1 \]

4. It has been shown that \( \chi(HF_*(M)) = \lambda_{CWL}(M) \), so we also have

\[ Gr(c_1(\mathcal{E})) = \lambda_{CWL}(M). \]

5. As a result,

\[ Gr(c_1(\mathcal{E})) = 0, \quad \text{if} \quad b_2^+(X) > 3. \]
New Math Results From Physics

• The Instanton and monopole Floer homologies of 3-manifolds.

1. Symplectic $X = M \times S^1$, where $M = \text{compact, closed 3-manifold and } b_1(M) > 1$.

2. $SW(X, \pi^{-1}(s_M)) = SW(M, s_M)$, where $\pi : M \times S^1 \to M$
New Math Results From Physics

3. \( \chi(HM_*(M, s_M)) = SW(M, s_M) \)

4. Altogether, via \( \text{Gr}(c_1(\mathcal{E})) = \chi(HF_*(M)) \). (pt. 2 in 3rd last slide) we have (up to a sign)

\[ \chi(HM_*(M, \pi(\mathcal{S}))) = \chi(HF^u_*(M)) \]

\[ c_1(\mathcal{S}) = c_1(\mathcal{E}) - \frac{1}{2} c_1(K), \text{ where } K \text{ is the canonical line bundle of } X, \text{ and } c_1(\mathcal{E}) \text{ is the Poincaré-dual of the fundamental class of the connected, non-multiply-covered pseudo-holomorphic curve } \Sigma \text{ with positive self-intersection.} \)
New Math Results From Physics

• For $M = \Sigma_g \times S^1$ with $g \geq 1$, result is a generalization of a conjecture by Kronheimer for $g = 0$.

• For $M = \Sigma_g \times S^1$ with $g \geq 1$, we have an isomorphism between $HF^w(M) \longleftrightarrow QH^*(\mathcal{M}_{\Sigma_g})$, and between $HM_*(M, \pi(\hat{S})) \longleftrightarrow QH^*(s^r(\Sigma_g))$.

• Cobordism construction smoothly linking $QH^*(\mathcal{M}_{\Sigma_g})$ and $QH^*(s^r(\Sigma_g))$. 
New Math Results From Physics

• An identity of the SW invariants of M.

1. Symplectic $X = M \times S^1$, where $M = \text{compact, closed 3-manifold}$ and $b_1(M) > 1$.

2. Via $\text{Gr}(c_1(\mathcal{E})) = \lambda_{CWL}(M)$, $SW(\hat{s}) = \pm \text{Gr}(c_1(\mathcal{E}))$, relation between Reidemeister-Milnor Torsion and $\lambda_{CWL}(M)$, and Meng-Taubes, (up to a sign)

$$SW(M, \pi(\hat{s})) = \sum_{x \in H} \sum_{s_M | c_1(s_M) = x} SW(M, s_M)$$
New Math Results From Physics

• For $M = \Sigma_g \times S^1$ with $g \geq 1$, this is consistent with all known math results.

• Due to time constraint, we will not discuss alternative and economical physical derivations of

1. Properties of knot-homology groups from singular instantons (Kronheimer-Mrowka)

2. Vanishing theorem of monopole Floer homology of $M$ (Kutluhan-Taubes)

3. Seiberg-Witten theory on symplectic 4-manifolds with $b_1 = 0$ and $b^+ > 1$. 
Conclusion

• Rich interplay between the physics of surface operators in supersymmetric topological gauge theory, and the geometry and topology of 4 and 3-manifolds.

• Much more can be done, especially on the physics side, which can potentially link the story to the geometric Langlands program, which involves Higgs bundles etc.