Moduli spaces of hyperbolic surfaces with cone angles.

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Summary.

- Define moduli spaces of hyperbolic surfaces with cone angles.
- These are equipped with symplectic forms and hence have well-defined volumes depending on the cone angles.
- Mirzakhani proved that volumes of moduli spaces of hyperbolic surfaces with geodesic boundary lengths are polynomial in the lengths.
- The volume polynomial analytically continues to give volumes of moduli spaces of hyperbolic surfaces with small cone angles.
- **Question:** how are Mirzakhani’s volume polynomials related to the volumes of moduli spaces of hyperbolic surfaces with large cone angles?
\[ \mathcal{M}_{g,n}(L_1, \ldots, L_n) = \text{moduli space of oriented hyperbolic surfaces with length } L_i \text{ geodesic boundary components.} \]

Model of a neighbourhood of a length \( L \) closed geodesic

\[
g_L(L^2) = \begin{pmatrix} \cosh \frac{L}{2} & \frac{2L}{L} \sinh \frac{L}{2} \\ \frac{L}{2} \sinh \frac{L}{2} & \cosh \frac{L}{2} \end{pmatrix}, \text{ fixed points } \pm 2 \sqrt{L^2} \in \mathbb{H}^2
\]

Generalise \( L^2 > 0 \) to \( L^2 \in \mathbb{R} \).

\( L^2 > 0 \)—closed geodesic, \( L^2 = 0 \)—cusp, \( g_L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)

\( L^2 < 0 \)—cone angle, \( g_L \) rotation by \( \phi \) for \( L = i\phi \)
\( \mathcal{M}_{g,n}(L_1, \ldots, L_n) = \) moduli space of oriented hyperbolic surfaces with geodesic boundary components, cusps and cone angles corresponding to \( L_j = i\phi_j \).

Different behaviours

- all \( L_j = 0 \) (cusps)
- all \( L_j = i\phi_j, \ 0 \leq \phi_j < 2\pi \)
- \( L_j > 0 \) or \( L_j = i\phi_j, \ \phi_j < \pi \)

Arc lengths \( \{a_i\} \) give (generalised) Penner coordinates.
Poisson structure on $\mathcal{M}_{g,n}(\cdot,\ldots,\cdot)$.

$$\eta_{WP} = \sum_{j=1}^{n} \sum_{k,l} \frac{\sinh(\alpha_{j,kl}L_j/2)}{\sinh(L_j/2)} \frac{\partial}{\partial a_k} \wedge \frac{\partial}{\partial a_l} \quad \text{(Mondello)}$$

$$\alpha_{j,kl} = 1 - 2 \times \text{(fraction of rotation around } L_j \text{ between arcs)}$$

$\eta_{WP}$ is degenerate—non-degenerate on $L_j = \text{constant}$

$\omega_{WP}$ dual Weil-Petersson symplectic form

$$V_{g,n}(L_1,\ldots,L_n) = \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \frac{\omega_{WP}^{3g-3+n}}{(3g-3+n)!}$$
**Theorem** (Mirzakhani) $V_{g,n}(L_1, \ldots, L_n)$ is polynomial in $L_i^2$.

Uses a McShane identity.

True for $L_j \geq 0$, $L_j = i\phi_j$, $\phi_j \leq \pi$. (Tan-Wong-Zhang)

**Q.** How is $V_{g,n}(L_1, \ldots, L_n)$ related to the volume of the moduli space for cone angles $> \pi$?

**Example.** $V_{0,4}(L_1, \ldots, L_4) = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2 + 4\pi^2)$ does not give the volume for large enough angles.

**Guess:** the polynomial gives the volume when there is only one cone angle ($< 2\pi$.)
**Theorem** (Norman Do, N.)

\( V_{g,n+1}(L_1, \ldots, L_n, 2\pi i) = \sum_{k=1}^{n} \int_{0}^{L_k} L_k V_{g,n}(L_1, \ldots, L_n) dL_k \)

\( \frac{\partial V_{g,n+1}}{\partial L_{n+1}}(L_1, \ldots, L_n, 2\pi i) = 2\pi i(2g - 2 + n) V_{g,n}(L_1, \ldots, L_n) \)

For \( 0 \leq \phi_j < 2\pi \) there exists a forgetful map

\[ \mathcal{M}_{g,n+1}(i\phi_1, \ldots, i\phi_n, i\phi_{n+1}) \rightarrow \mathcal{M}_{g,n}(i\phi_1, \ldots, i\phi_n). \]

As \( \phi_{n+1} \rightarrow 2\pi \) the Kähler metric degenerates along fibres and tends to the pull-back of the Kähler metric downstairs. (Schumacher-Trapani)

Specialise (1) to

\( V_{g,n+1}(0, \ldots, 0, 2\pi i) = 0. \)
Study the degeneration as $\phi_{n+1} \to 2\pi$.

$$\sin \frac{\phi_{n+1}}{2} \cdot \eta_{WP} \to \sum_{k,l} \sin(\phi_{n+1,kl}) \frac{\partial}{\partial a_k} \wedge \frac{\partial}{\partial a_l}.$$ 

Elementary geometry.

$$\{a_i, a_j\} = \sin \phi_{ij}$$

Lengths $a_i$ are functions on the hyperbolic surface. Hyperbolic metric (Kähler) gives Poisson structure $\eta_{hyp}$. 
The uniform convergence

\[ \sin \frac{\phi}{2} \cdot \eta_{WP} \to \eta_{hyp} \]

almost gives (3) and (2).

Idea

\[ \omega_{WP,g,n+1} \sim \omega_{WP,g,n} + \sin \frac{\phi}{2} \cdot \omega_{hyp} \]

For \( N = 3g - 3 + n \),

\[ \frac{\omega_{WP,g,n+1}^{N+1}}{(N + 1)!} \sim (N + 1) \frac{\omega_{WP,g,n}^N}{(N + 1)!} \cdot \sin \frac{\phi}{2} \cdot \omega_{hyp} \]

which should integrate to give

\[ \text{Vol}_{g,n+1} \sim 4\pi (2g - 2 + n) \sin \frac{\phi}{2} \cdot \text{Vol}_{g,n}(L_1, \ldots, L_n). \]
Eynard and Orantin also (rigorously) prove (1) and (2).

- A model / B model mirror picture

- A model side: $V_{g,n}(L_1, \ldots, L_n)$—generating function for Gromov-Witten invariants with Kähler parameters as variables.

- B model side: Laplace transform of $V_{g,n}(L_1, \ldots, L_n)$
  - Underlying the B model is a Riemann surface $\Sigma$ equipped with a meromorphic 1-form $\theta$ and a map $\Sigma \to S^2$.
  - B model is concerned with variations of periods of $\theta$.
  - Equations (1) and (2) are special cases of general properties of the B model.