A Variation of McShane's Identity
for 2-bridge links

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Plan

1. Farey tessellation and 2-bridge links
2. Punctured torus groups and 2-bridge link groups
3. McShane’s identity and its variation
4. Word, Conjugacy, Peripheral problems for 2-bridge link groups
5. Relation with Minsky’s question
6. Tan-Wong-Zhang’s end invariants
\[ S = \mathbb{R}^2 - \mathbb{Z}^2 / \langle \pi \text{-rotations around punctures} \rangle : 4 \text{-punctured sphere} \]

(Conway sphere)

\[ \Rightarrow \]

\[ \delta \approx \delta \]

\[ D \text{: Farey tessellation} \]

\[ \begin{align*}
&\text{Vertex set of } D = \hat{Q} = \mathbb{Q} \cup \{ \frac{1}{0} \} \approx \mathbb{R} \\
&\quad \leftrightarrow \{ \text{essential simple loops on } S \} = \alpha_r \\
&\quad \leftrightarrow \{ \text{essential simple arcs on } S \} = \mathcal{S}_r \\
&\text{Farey triangle} \quad \leftrightarrow \quad \text{ideal triangulation of } S
\end{align*} \]
$S := \quad \iff S^2 - 4 \text{ points} \quad \implies T$

$S$ and $T$ are "commensurable"
Rational tangle \((B^3, t(n))\) of slope \(r\): 

\[
\pi_1(B^3 - t(r)) = \pi_1(S)/\langle dr \rangle
\]

\((S^3, K(n)) = (B^3, t(\omega)) \cup (B^3, t(n))\): 2-bridge link of slope \(r\)

\[G_t(K(n)) := \pi_1(S^3 - K(n)) = \pi_1(S)/\langle d\omega, dr \rangle\]
\[ \cdot \text{K}(r) \text{ is hyperbolic, i.e. } S^3 - \text{K}(r) \text{ admits a complete hyperbolic structure of finite volume} \]

\[ \Leftrightarrow \ b \not\equiv 1 \mod p, \text{ where } r = \frac{b}{p} \Leftrightarrow d(\infty, r) \geq 3 \]

\[ \cdot \varphi : \pi_1(S) \rightarrow \pi_1(S) \]

\[ \left\langle d_\infty, d_r \right\rangle \cong G_r(K(r)) \hookrightarrow \text{PSL}(2, \mathbb{C}) \]

\[ \downarrow \]

\[ \varphi_r : \pi_1(T) \rightarrow \text{PSL}(2, \mathbb{C}) \]

where \( T := \mathbb{R}^2 - \mathbb{Z}^2 / \mathbb{Z}^2 \) is a punctured torus

\[ (\therefore) \quad T \text{ and } S \text{ are "commensurable"} \]

\[ \cdot \varphi_r \text{ is a discrete non-faithful representation of } \pi_1(T) \text{ or } \pi_1(S) \]
Conjectural characterization of the 2-bridge Kleinian groups

\[ \mathcal{R} := \{ p : \pi_1(T) \to PSL(2, \mathbb{C}) \mid \text{type-preserving} \} \]

\[ \mathcal{D} := \{ \text{discrete faithful} \} = \mathcal{Q}^\text{f} \cong \mathbb{H}^2 \times \mathbb{H}^2 - \text{diag}(\mathbb{H}^2) \]

\[ \mathcal{Q}^\text{f} := \{ \text{quasi-Fuchsian} \} \cong \mathbb{H}^2 \times \mathbb{H}^2 \]

\[ \mathcal{F} := \{ \text{Fuchsian} \} \cong \text{diagonal } (\mathbb{H}^2 \times \mathbb{H}^2) \cong \mathbb{H}^2 \]

\[ \mathcal{D}_{\text{finite}} := \{ \text{discrete (non-faithful) co-finite volume} \} \]

\[ \{ p \mid r = \frac{3}{p}, \ g \# z \in (p) \} = \{ 2\text{-bridge Kleinian group} \} \]
[Akiyoshi - S - Wada - Yamashita]

There is a natural "path" \( \{ Pr, \theta_1, \theta_2 \mid 0 \leq \theta_i \leq 2\pi \} \)
from \( Q7i \) to \( Pr \)

[Conjecture]

\( p \in R \) is discrete iff \( p \in Q7 = \overline{Q7} \)

or \( p = Pr, 2\pi/m, 2\pi/n \)
McShane's identity. For any $p \in \mathbb{F}$,

$$\sum_{s \in \mathbb{D}} \frac{1}{1 + \exp(l(p(x_s)))} = \frac{1}{2}$$

where $l(p(x_s))$ = translation length of $p(x_s)$

McShane, Tan-Wong-Zhang, Mirzakhani

Generalization to higher genus surfaces with punctures, boundaries, or cone-points.

Application to Weil-Petersson volume of moduli spaces

Bowditch, Akiyoshi-Miyachi-S, Tan-Wong-Zhang

3-dimensional variation for $p \in \mathbb{Q}_\mathbb{F} = \mathbb{D}$
Theorem (A variation of McShane's identity for 2-bridge knots)

For \( \varphi : \text{Teich}(T) \to \text{PSL}(2, \mathbb{C}) \) corresponding to the complete hyperbolic structure of \( S^3 - K(r) \):

\[
2 \sum_{s \in I_1} \frac{1}{1 + \exp(l(\varphi(s)))} + \sum_{s \in \partial I_1} \frac{1}{1 + \exp(l(\varphi(s)))} = -1 - 2 \sum_{s \in \partial I_2} \frac{1}{1 + \exp(l(\varphi(s)))} - \sum_{s \in \partial I_2} \frac{1}{1 + \exp(l(\varphi(s)))} = \text{Modulus of the cusp torus with a suitable choice of the longitude.}
\]
(Outline of proof)

(1) The series converges absolutely.
   • A refinement of Bowditch's work by Akiyoshi-Miyachi-S and Tan-Wong-Zheng.
   • Discreteness of the length spectrum of finite volume hyperbolic manifolds

○ Explicit solution of some special word problem, conjugacy problem, and peripheral problem for 2-bridge knot groups

(2) The series represents the modulus of the cusp torus
   S-Weeks decomposition of 2-bridge link complements, which are shown to be geometric by [Futer-Gueritaud] [ASWY]
[Schubert] \( K(r) \cong K(r') \)

\[ \iff \text{The relative positions of } \{\infty, r\} \text{ and } \{\infty, r'\} \text{ in } D \text{ are equivalent} \]

\[ \exists \delta \in \text{Aut}(D) \text{ st } \delta \{\infty, r\} = \{\infty, r'\} \]

**Def.** The group \( \hat{\Gamma}_r \) associated with \( K(r) \)

\[ \Gamma_r := \langle \text{reflections in the edges of } D \text{ with an endpoint } r \rangle \subset \text{Aut}(D) \]

\[ \cong \text{infinite dihedral group } D_{\infty} \]

\[ \hat{\Gamma}_r := \langle \Gamma_\infty, \Gamma_r \rangle \cong \Gamma_\infty \times \Gamma_r \text{ if } d(\infty, r) \geq 2 \]
\[ \alpha_s \sim \alpha_{s'}, \text{ in } B^3 - t(r) \iff S' = \tau(s) \text{ for some } \tau \in \Gamma_r \]

(proof)

(\Leftarrow) We may assume \( r = \infty \) and \( \tau \) is a reflection in \( \langle \infty, 0 \rangle \)

Then \( \tau \) is induced by \( f : (B^3, t(\infty)) \to (B^3, t(\infty)) \),
\[ \text{i.e. } f(\alpha_s) = \alpha_{-s} = \alpha_{\tau(s)} \]

On the other hand, \( f_* = \text{id} : \Pi_i (B^3 - t(\infty)) \to \Pi_i (B^3 - t(\infty)) \).

Hence \( \alpha_{\tau(s)} \sim f(\alpha_s) \sim \alpha_s \text{ in } B^3 - t(\infty) \). \qed
Prop A (Ohtsuki - Riley - S)
\[ \alpha_s \sim \alpha_{s'} \text{ in } S^3 - K(r), \text{ if } S' = \tau(s) \text{ for some } \tau \in \hat{P}_r = P_{\infty} \ast P_r. \]
(proof)
If \( \tau \in \hat{P}_\infty \), then \( \alpha_s \sim \alpha_{\tau(s)} \) in \( B^3 - t(\infty) \) and hence in \( S^3 - K(r) \) because \( S^3 - K(r) = (B^3 - t(\infty)) \cup (B^3 - t(r)) \).
Similarly, if \( \tau \in P_r \), then \( \alpha_s \sim \alpha_{\tau(s)} \) in \( B^3 - t(r) \), hence in \( S^3 - K(r) \). \( \square \)

Cor A (ORS)
\[ \alpha_s \sim 1 \text{ in } S^3 - K(r), \text{ if } S \in \hat{P}_r \cdot \infty \cup \hat{P}_r \cdot r. \]
(proof)
If \( S = \tau(\infty) \) for some \( \tau \in \hat{P}_r \), then \( \alpha_s \sim \alpha_{\infty} = 1 \) in \( S^3 - K(r) \)
\( (S = \tau(r)) \)
\( (\alpha_s \sim \alpha_r = 1) \) \( \square \)
Question Does the converse to Cor A hold?

Which simple loop on the level 4-punctured sphere $S$ in a 2-bridge knot complement $E(K(r)) = S^3 - K(r)$ is null-homotopic in $E(K(r))$?

Main Theorem 1 [Donghi Lee - S]

The converse to Cor A holds.

Namely, $\alpha_S = 1$ in $G_i(K(r))$ iff $s \in \hat{P}_r \{\infty, r\}$.

(Idea)

Small cancellation theory
Lemma  For any \( S \in \hat{Q} \), there is a unique \( S_0 \in I_1 \cup I_2 \) such that \( S \in \hat{\pi}_r S_0 \).

(Proof) This follows from the fact that \( R \subset H^2 \) is a fundamental domain for the action \( \hat{\pi}_r \cap H^2 \).

Thus Main Theorem 1 is equivalent to:

Theorem 1' If \( S \in I_1 \cup I_2 \), then \( \alpha_S \neq 1 \) in \( G(K(r)) \).
Upper presentation of $G_t(K(r)) = \frac{\pi_1(S)}{\langle \alpha_\infty, \alpha_r \rangle} = \frac{\pi_1(B^3 - t(\infty))}{\langle \alpha_r \rangle}$

$\pi_1(B^3 - t(\infty)) = \langle x, y \rangle$

For a loop $\alpha \subset S$,

$[\alpha] \in \pi_1(B^3 - t(\infty)) = \langle x, y \rangle$

is obtained by "reading" the intersections of $\alpha$ with $\beta_x$ and $\beta_y$. 
Example

\[ (W_{\tilde{y}} := y \times \tilde{y} \times \tilde{y}) \]
\[ \alpha_{\tilde{x}} = x \cdot W_{\tilde{y}} \cdot \tilde{y} \cdot W_{\tilde{z}} \]
\[ = x(y \times \tilde{y} \times \tilde{y}) \times (z \times \tilde{y} \times \tilde{y}) \]

For \( 0 < \frac{3}{p} < 1 \)

\[ W_{\tilde{y}} = y \cdot x \cdot \tilde{x} \cdot y \cdot \tilde{x} \ldots \bigcirc_{\tilde{x}^{p-1}} \quad \tilde{x} = (-1)^{\left[ \frac{3}{p} \right]} \]

\[ \alpha_{\tilde{y}} = x \cdot W_{\tilde{y}} \square^{(-1)^{\tilde{x}}} \cdot W_{\tilde{y}} \quad (\text{length} = 2 \cdot \tilde{p}) \]

Note: \( x^{\tilde{x}} \) and \( y^{\tilde{x}} \) appear alternatively in \( \alpha_{\tilde{y}} \)

ie \( x^{\tilde{x}} \) nor \( y^{\tilde{x}} \) does not appear in \( \alpha_{\tilde{y}} \)
Key Observation: \( \frac{2}{5} = \frac{1}{3} \oplus \frac{1}{2} \) (Farey neighbors)

\[
\alpha_{\frac{2}{5}} = \frac{1}{v_1} \cdot \frac{1}{v_2} \cdot \frac{1}{v_3} \cdot \frac{1}{v_4} \quad v_1, v_3 \leftrightarrow \frac{1}{3}
\]

\[
\frac{2}{P} = \frac{\theta_1}{P_1} \oplus \frac{\theta_2}{P_2} \quad \text{with} \quad \frac{\theta_1}{P_1} < \frac{\theta_2}{P_2}, \quad \text{then}
\]

\[
\alpha_{\frac{2}{P}} = \frac{1}{v_1} \cdot \frac{1}{v_2} \cdot \frac{1}{v_3} \cdot \frac{1}{v_4} \quad \text{where} \quad v_1, v_3 \leftrightarrow \frac{\theta_1}{P_1} \quad \text{and} \quad v_2, v_4 \leftrightarrow \frac{\theta_2}{P_2}
\]
(Outline of the proof of Theorem 1') \( U = \alpha r \)

1. The presentation \( G(\text{K}(r)) = \langle x, y | U \rangle \), where \( U = \alpha r \), satisfies the conditions C(4) and T(4) in the small cancellation theory.

2. If \( w \) is a reduced word in \( \langle x, y \rangle \), st
   \( \text{cyclically} \)
   (i) \([w] = 1 \) in \( G(\text{K}(r)) = \langle x, y | U \rangle \)

   (ii) \( x^{\pm 2} \) nor \( y^{\pm 2} \) is a subword of \( w \), i.e., \( w \) is "alternating". Then \( w \) contains some "special" subword \( w_0 \), st

   \( w_0 \) is a subword of the cyclic word \( U \) or \( U^{-1} \)

   with \( |w_0| > \frac{1}{2} |U| \)

(proof) Small cancellation theory
(3) If $s \in I_1 \cup I_2$, then the cyclic word representing $\alpha_s$ does not satisfy the condition in (2). Hence $\alpha_s \neq 1$ in $G(K(r))$. ⊲

(Intuition behind (3))

$S \in I_1 \cup I_2$

$\Leftrightarrow$ The slope $S$ is "far from" $\infty$ and $r$

$\Leftrightarrow$ $\alpha_s$ and $u = \alpha_r$ cannot share a long subword.
Minsky's Question (Geom. Top. Monograph 12)

- $V$: handlebody
  
  $$m(\partial V) := \pi_0 \text{Diff}(\partial V) \cup m(V) := \pi_0 \text{Diff}(V) \cup m_0(V) := \{f \in m(V) | f_* = \text{id} \in \text{Out}(\pi_1(V))$$

- $M = V_+ \cup S \cup V_-$ Heegaard splitting

$$\Gamma_\pm := m_0(V_\pm) \subset m(S) \leftrightarrow \Gamma_\pm, \hat{\Gamma}_\pm$$

$$\Gamma := \langle \Gamma_+, \Gamma_- \rangle \subset m(S) \leftrightarrow \hat{\Gamma}_r := \langle \hat{\Gamma}_\pm \rangle$$

$$\Delta_\pm := \{\text{meridians of } V_\pm\} \subset \mathcal{C}(S) \leftrightarrow \{\alpha\}, \{\gamma\}$$

**Fact** If $\alpha \in \Gamma(\Delta_+ \cup \Delta_-)$, then $\alpha = 1$ in $\pi_1(M)$. \leftrightarrow [ORS]

**Question** Does the converse hold when $M$ is hyperbolic?

Main Theorem 1 may be regarded as an answer to a special variation of the above question.
Question: For $s, s' \in I_1 \cup I_2$, when $\alpha_s \sim \alpha_{s'}$ in $S^3 - K(r)$?

Theorem 2 (Conjugacy problem)

1. Suppose $r = \frac{1}{p}$, i.e. $d(\infty, r) = 2$.
   For $s, s' \in I_1 \cup I_2$,
   $\alpha_s \sim \alpha_{s'}$ in $S^3 - K(r)$
   iff $s' = s$ or $\gamma(s)$.

2. Suppose $p \not\equiv \pm 1 \pmod{p}$, i.e. $d(\infty, r) \geq 3$.
   Then for $s \neq s' \in I_1 \cup I_2$,
   $\alpha_s + \alpha_{s'}$ in $S^3 - K(r)$. 
Geometric proof of the if part of (1)

(i) \( \Phi \) interchanges \((B^3, t_{\infty})\) with \((B^3, t_{\frac{1}{\Phi}})\), and the action of \( \Phi \) on the Farey tessellation is equal to \( \Phi \)

i.e \( \Phi(\alpha_s) = \alpha_s \).

(ii) \( \Phi^* \in \text{Aut}(\text{Gr}(K(\frac{1}{\Phi}))) \) is given by \( \Phi^*(x) = x^{-1}, \Phi^*(y) = y^{-1} \).

This implies \( \Phi^*(\alpha_s) \) is conjugate to \( \alpha_s^{\pm 1} \) in \( \text{Gr}(K(\frac{1}{\Phi})) \).

Hence \( \alpha_{\Phi}(s) = \Phi(\alpha_s) \sim \alpha_s^{\pm 1} \) in \( S^3 - K(\frac{1}{\Phi}) \).
Theorem 3 (Peripheral problem) Assume \( g \neq 2l(p) \), i.e. \( d(\infty, r) \geq 3 \).

1. Suppose \( K(r) \not\cong K\left(\frac{m}{2n+1}\right) \ (n \in \mathbb{Z}) \),
then any \( \alpha_S \ (S \in I_1 \cup I_2) \) is not peripheral.

2. Suppose \( K(r) \cong K\left(\frac{m}{2n+1}\right) \), then
\[
\# \{ S \in I_1 \cup I_2 \mid \alpha_S \text{ is peripheral} \} < +\infty
\]

Remark \( \alpha_{\frac{n+1}{2n+1}} \) is peripheral in \( \mathbb{S}^3 - K\left(\frac{m}{2n+1}\right) \).

The clasp disk gives a homotopy between \( \alpha_{\frac{n+1}{2n+1}} \) and a peripheral loop.
Tan-Wong-Zhang's end invariants $E(p)$

For $p: \pi_1(T) \to SL(2,\mathbb{C})$, set

$$E(p) = \left\{ \lambda \in \hat{\mathbb{R}} \mid \exists K > 0, \exists \{ r_n \} \subset \hat{\mathbb{Q}}, \text{ st } r_n \to \lambda, |\text{tr} \ p(a_m)| < K \right\}$$

**Theorem.** Let $p_r: \pi_1(T) \to PSL(2,\mathbb{C})$ be the discrete non-faithful representation corresponding to the complete hyperbolic structure of $S^3 - K(r)$. Then

$$E(p_r) = \Lambda(\hat{p}_r) : \text{the limit set of } \hat{p}_r \equiv \text{Cantor set}$$

**Question.** Does the converse hold?

i.e. if $E(p) = \Lambda(\hat{p}_r)$, then is $p$ conjugate to $p_r$?
**Question**

Let \( \rho, \theta_1, \theta_2 : \text{Ti}(T) \to \text{PSL}(2, \mathbb{C}) \) be the representation corresponding to the hyperbolic cone-manifold.

Then

- \( \varepsilon(\rho, \theta_1, \theta_2) \cap (I_1 \cup I_2) = \emptyset \) ?

- A variation of McShane's identity holds for \( \rho, \theta_1, \theta_2 \)?