Affine deformations of the three-holed sphere and other surfaces

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We will look at groups $G$ of affine transformations of $\mathbb{A}^3$, affine three-space, whose linear parts are the holonomy of a hyperbolic structure on a surface with boundary. We will consider the following question: when does $G$ act freely and properly discontinuously on $\mathbb{A}^3$? This is joint work with Todd Drumm and Bill Goldman.
Introduction

Affine deformations

Ideal triangle configurations

The three-holed sphere

The one-holed torus
A little history

- Milnor’s question: Given a manifold $M = \mathbb{R}^3/G$, where $G$ consists of affine transformations, must $G$ be solvable?
- The alternative to Milnor’s question: can a free group of affine transformations act freely and properly discontinuously on $\mathbb{A}^3$?
- Margulis’ answer (to the alternative): yes - take a Schottky group and add appropriate translational parts to generators.
- Fried-Goldman: If $G$ is not solvable, then taking its linear part embeds it as a (conjugate of a) discrete subgroup of $O(2, 1)$.
- Mess: In that case, the linear part of $G$ is not the holonomy of a closed surface.
Notation and terminology

- \( \mathbb{R}_1^3 = \mathbb{R}^3 \) with symmetric indefinite bilinear form of signature (2, 1):
  \[ \langle x, y \rangle = x_1y_1 + x_2y_2 - x_3y_3. \]
- \( A_1^3 \) is the affine space modeled on \( \mathbb{R}_1^3 \)
- \( \mathbb{H}^2 \) denotes the hyperbolic plane; think \( \mathbb{H}^2 \subset \mathbb{R}_1^3 \).
Cocycles and affine deformations

From now on: $\Gamma < \text{SO}(2,1)^0$ is a free group.

A cocycle $u \in Z^1(\Gamma, \mathbb{R}_1^3)$ yields a representation $\phi_u : \Gamma \to \text{Aff}(\mathbb{A}_1^3)$, by setting:

$$\phi_u(g) : p \mapsto g(p - o) + u(g),$$

where $o$ is a choice of origin.

Call $\phi_u(\Gamma)$ an affine deformation of $\Gamma$, and a proper affine deformation if it acts (freely and) properly discontinuously on $\mathbb{A}_1^3$. 
The Margulis invariant

The *Margulis invariant* of $u$, denoted $\alpha_u$, is the linear functional:

$$\alpha_u(g) = \langle u(g), x^0(g) \rangle$$

where $x^0(g)$ is a preferred fixed eigenvector of $g$. For hyperbolic $g$, it is the *signed Lorentzian displacement* along a $\phi_u(g)$-invariant line in $\mathbb{A}^3_1$. 
The Margulis invariant and cohomology

For a rank 2 free subgroup
\[ \Gamma = \langle g_1, g_2, g_3 \mid g_3 g_2 g_1 = Id \rangle < \text{SO}(2, 1)^0, \quad H^1(\Gamma, \mathbb{R}^3) \] is parametrized using the Margulis invariant:

\[ [u] = (\alpha_u(g_1), \alpha_u(g_2), \alpha_u(g_3)) \]

(Drumm-Goldman)
Proper affine deformations lie in an octant of $H^1(\Gamma, \mathbb{R}^3_1)$

Set:

$$\mathcal{H}_g = \{ [u] \in H^1(\Gamma, \mathbb{R}^3_1) \mid \alpha_u(g) > 0 \}$$

Then the set of proper affine deformations is contained in

$$\bigcap_{g \in \Gamma} \mathcal{H}_g$$

(Margulis+absence of loss of generality)
Ideal triangle configurations

**The basic idea:** Geodesics in $\mathbb{H}^2$ correspond to *crooked planes* in $\mathbb{A}_1^3$.

Our strategy will be to move the crooked planes away from each other along the edges of an ideal triangulation to obtain fundamental domains for proper action.

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★ We want pairwise disjoint crooked planes in order to apply a Klein-Maskit type combination theorem.
What you need to know about crooked planes for this talk

In a world without a positive definite metric, where discrete groups may not act properly, you need something like crooked planes to build fundamental domains.
Ideal triangle configurations and the three-holed sphere

Let $\Sigma = \mathbb{H}^2 / \Gamma$ be a three-holed sphere. Write:

$$\Gamma = \langle g_1, g_2, g_3 \mid g_3 g_2 g_1 = \text{Id} \rangle$$

Let $T$ be the ideal triangle whose vertices are the attracting fixed points of the $g_i$.

$T$ yields half of an ideal triangulation of the interior of $\Sigma$’s convex core.
With a little work, one obtains proper affine deformations by moving crooked planes “along the edges of \( T \).
Such a move will look something like this for a quadruple of crooked planes.

\[
\text{Crooked plane stems paired by } g_1 \\
\text{Crooked plane stems paired by } g_2 \\
\text{Invariant axes of generators}
\]
For the three-holed sphere, this yields all of $\mathcal{H}_{g_1} \cap \mathcal{H}_{g_2} \cap \mathcal{H}_{g_3}$

**Theorem (C-Drumm-Goldman)**

Let $\Gamma = \langle g_1, g_2, g_3 \mid g_3g_2g_1 = \text{Id} \rangle < \text{SO}(2,1)^0$, which is the holonomy of a three-holed sphere, and let $u \in Z^1(\Gamma, \mathbb{R}_1^3)$; suppose that $\alpha_u(g_i)$, $i = 1, 2, 3$, are all of the same sign. Then $\phi_u(\Gamma)$ admits a fundamental domain. In particular, it acts freely and properly discontinuously on $\mathbb{A}_1^3$.

(Compare Jones and Goldman-Labourie-Margulis.)
Ideal triangle configurations for the one-holed torus

Let $\Gamma = \langle g_1, g_2, g_3 \mid g_1 g_2 g_3 = ld \rangle$ now be the holonomy of a one-holed torus.

We consider an ideal triangulation with vertices judiciously chosen amongst the fixed points of the commutators.

We obtain various regions of proper cocycles by *changing the generating set*:

$$(g_1, g_2, g_3) \mapsto (g_2^{-1}, g_1, g_1^{-1} g_2)$$
Result for the one-holed torus

Theorem (C-Drumm-Goldman)

Let $\Gamma$ be the holonomy of a one-holed torus. The set of cohomology classes of cocycles admitting a fundamental domain is the interior of the set:

$$\bigcap_{g \in SCC} \mathcal{H}_g$$

where $SCC \subset \Gamma$ is the set of elements corresponding to simple closed curves.