Understanding Weil–Petersson curvature

Moduli of flat tori

Fenchel–Nielsen description
given lengths $l_1$, $l_2$, $l_3$

right hexagon $l_{1/2}$ $l_{2/2}$ $l_{3/2}$

double across alternating $l_2$

reflection $l_1$

$T$: hyperbolic displacement foot points initially measured, then develop to $R$-valued

Theorem FN $(l_i, r_j) \in (R_+ \times R)^{3g-3}$ are real analytic coordinates for Teichmüller space: marked hyperbolic structures

Construction provides homeomorphisms between surfaces

Teichmüller space: $\mathbb{S} (R, (C^+)^3$

$\text{hyperbolic structure homotopy class of }$ homeo of homeo from $F$

$\text{MCG} = \text{Homeo}(F)/\text{Homeo}_0(F)$ acts by pre composition
Construction provides for Baily-Borel type partial compactification

Lengths $\ell_j = 0$ describe degenerate hyperbolic structures

Augmented Teichmüller space $\bar{\mathcal{T}}$; for each homotopy distinct pants decomposition - allow lengths $\ell = 0$ with rate when $\ell = 0$ then $\bar{\mathcal{T}}$ is undefined - nhbds determined by mapping to small lengths and not small lengths & twists is continuous

Points of $\bar{\mathcal{T}} - \mathcal{T}$ describe degenerate structures

$\bar{\mathcal{T}}$ not locally compact, $\bar{\mathcal{T}}/\text{MCG}$ is homeomorphic to Deligne-Humphford stable curve compactification of moduli space

Theorem. $(\mathcal{Y}, \text{Daskalopoulos-Wentworth}, \omega)$ $\mathcal{Y}$ is CAT(0) metric space w/ zero level sets $\Sigma \ell_j = \ldots = \ell_{j+2} = 0$ geodesically convex

Geodesic-length functions $l_\alpha, l_\beta, l_\gamma$ provide a collection of functions for understanding geometry of $\bar{\mathcal{T}}$ & hyperbolic structures
cokantant space $\mathcal{Q}(R)$ holomorphic quadratic diff's on $R$

WP Hermitian metric $\langle \phi, \psi \rangle = \int_R \phi \overline{\psi} (dA_{\text{hyp}})^{-1}$

Program to describe metric in terms of geodesic-length functions

Riera $\langle \text{grad} l_\alpha, \text{grad} l_\beta \rangle = \frac{8}{\pi} \left( S_{\alpha \beta} + \sum \frac{u \log |\frac{u+1}{u-1}| - 2}{\alpha^1 \Gamma(\beta)} \right)$

where for lifts $\tilde{\alpha}, \tilde{\beta}$, $u(\tilde{\alpha}, \tilde{\beta}) =$ \begin{cases} \cos \theta & \text{if } \tilde{\alpha} \wedge \tilde{\beta} \neq \phi \\ (\cosh d(\tilde{\alpha}, \tilde{\beta})) & \text{otherwise} \end{cases}

for $d$ large, summand $u \frac{e^{-2d}}{3}$

Definition for a geodesic-length set $l_\alpha = \text{grad} l_\alpha^{1/2}$

Find $\langle l_\alpha, l_\beta \rangle = (\cosh)^{-1} S_{\alpha \beta} + O(l_\alpha^{3/2} l_\beta^{3/2})$ for $l_\alpha, l_\beta$ bounded

$\omega$ expansions for Hessian of $l_\alpha$, Levi-Civita connection and Riemann tensor

$D_{\alpha} l_\alpha = 3 l_\alpha^{1/3} \langle \mathcal{L} l_\alpha, u \rangle T_l l_\alpha + O(l_\alpha^{3/2} l\text{null})$

$l_\alpha$ bounded

$T$ complex structure
for Riemann C-tensor
\[ R(\lambda_x, \lambda_x, \lambda_x, \lambda_x) = \frac{3 \langle \lambda_x, \lambda_x \rangle}{4\pi \ell_x} + O(\ell_x^{-1}) \]
\[ \ell_x \text{ bounded, most pairs coinciding} \]

\[ R(\lambda_x, \lambda_y, \lambda_y, \lambda_y) = O((\ell_x \ell_y \ell_y \ell_y)^{\frac{1}{2}}) \]
\[ \ell_x \text{ bounded, most pairs coinciding} \]

Metric expansion near
\[ \sum_{g} \gamma_g \]
\[ \text{S partial compactification} \]

\[ \langle \ , \ \rangle_{wp} = 2\pi \sum_{\text{small lengths}} (\ell_x^{\frac{1}{2}})^2 + (\ell_x^{\frac{1}{2}} \circ J)^2 \]
\[ + \langle \ , \ \rangle_{wp} / S + O(\ell_x^{\frac{1}{2}}(\ , \ )) \]

Introduce alternative to FN angle
\[ \rho_{\alpha} = \frac{2\pi}{\ell_x^{\frac{1}{2}} \langle \lambda_x, \lambda_x \rangle} \langle \ , \ J \rangle \]
(Does not involve pants decomposition; for \( \ell_x = c \) agrees with FN)

Introduce formal variable
\[ 2\pi \ell_x^2 \rho_{\alpha}^2 = \ell_x \]
\[ 2\pi(\ell_x^{\frac{1}{2}})^2 + (\ell_x^{\frac{1}{2}} \circ J)^2 = \pi^3 (4d^2 + r_0 \rho_{\alpha}^2)(1 + O(r_0^2)) \]
\[ 4d^2 + r_0^2 \theta^2 \]

Geometry near the partial compact stratum S
WP metric & connection & curvature \( \omega \)
Orthogonal product of \( 4d^2 + r_0^2 \theta^2 \) and
Lower dim! WP along S
5. Applications of Formulas

For $\Lambda(R)$ systole - length of shortest closed geodesic:

Sectional curvature at $R > \frac{-3-\varepsilon}{\pi R(R)}$

Holomorphic sectional curvature $\lambda \alpha = \frac{-3}{\pi L_\alpha} + O(L_\alpha)$

Product structure along $\mathcal{G}(\mathcal{G})$

$\pi \text{ span } \frac{\varepsilon}{\pi} \lambda \alpha^{\frac{3}{2}} \times \pi \text{ components } T^{1,0} g(R^g)$

Small lengths $\leq \lambda \alpha^{\frac{1}{2}}$

Deep surface $\neq (0, 3)$

A limit of tangent sections with curvatures tending to zero is a section with at most 1-d projection to a factor maximal dimension of asymptotic flat is # factors

Liu-Sun-Yau The following metrics are bi Lipschitz on $\mathcal{G}$

Teichmüller-Kobayashi, Kähler-Einstein, Bergman
Carathéodory, McHaleen Kähler hyperbolic, -WP Ricci form
and asymptotic Poincaré

LSY Despite the singularities of WP on Deligne-Mumford $\mathcal{M}$ the Chern classes computed with WP represent the appropriate bundles

LSY The pair $(\mathcal{M}, \mathcal{A})$ is infinitesimally rigid

Barns-Masur-Wilkinson WP geodesic flow on $T, \mathcal{M}$ (after removing incomplete orbits) is ergodic

Cavendish-Partier $C \leq \lim \frac{\text{diam}(\mathcal{M})}{g \sqrt{g \log g}} \leq \lim \frac{\text{diam}(\mathcal{M})}{\sqrt{g \log g}} \leq C'$
Distant-sum estimate

\[ \text{grad} \log \langle \Delta x, \Delta x \rangle, \ \mathcal{R}(\Delta x, \Delta y, \Delta z, \Delta \tilde{w}) \] and

\[ (\mathcal{L} - 2\mathcal{L})^{-1} \]

linearization of prescribed curvature operator

each has form

\[ \sum_{\text{A} \in \mathcal{P} \setminus \text{A}'} e^{-2d(\mathcal{L}, A_{\mathcal{P}})} \quad \text{or} \quad \sum_{\text{A} \in \mathcal{P} \setminus \text{A}'} e^{-2d(\mathcal{L}, A_{\mathcal{P}}')} \]

analysis of sums

- in \( \mathcal{H} \) \( e^{-sd(t)} \), \( s > 0 \), satisfies mean value estimate

\[ (u_{\mathcal{P}}) \leq c \int_{u_{\mathcal{P}}(r)} \nabla d \text{d} A \]

- enhanced collar/cusp lemma

\[ \text{osc}_r \leq \text{inj rad}(p) e^{\text{dep,collar}} \leq c_2 \]

- multiplicity \( \mathcal{H} \cap \mathcal{N} \) \( \mathcal{H}/\Gamma \) is overlap #

For \( \mathcal{A}(\mathcal{B}(p, r)) \), \( \mathcal{A} \in \Gamma \) overlaps given by a cyclic subgroup

have multiplicity \( \mathcal{A} \) \( \mathcal{(\text{inj radius})}^{-1} \) \( e^{\text{dep, collar/cusp}} \)
\[ \left| \sum_{A} n(A_{cp}) \right| \leq \sum_{A} \left| \mu_{A} \right| dA \leq e^{d_{cp,2collar}} \int_{A(B_{cp,r})}^{U A(B_{cp,r})} \left| \mu_{A} \right| dA \]

have for \( q \) a basepoint \( s = d_{cp}(q, \cdot) \)

\[ u \mapsto e^{-d_{cp}(q,p)} \quad dA = e^{S} d\Theta ds \]

suppose \( \bigcup_{A} A(B_{cp,r}) \subset B(q, \delta_{0}) \supset \)

right-handed integral \[ \int_{\delta_{0}}^{\infty} e^{-\delta S} e^{S} ds \]

\[ \leq e^{d_{cp,2collar}} e^{-d_{cp}(q,p)} \]

geometry: for surfaces with short geodesics (close to degenerate surfaces) final expression is small for single & double coset sums with identity coset removed.