Delambre-Gauss Formulas in Hyperbolic 4-Space

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1. Trigonometric formulas for spherical and hyperbolic triangles

**Spherical triangles.** Consider a spherical triangle in the unit sphere having side-lengths \(a, b, c \in (0, \pi)\) and corresponding opposite interior angles \(\alpha, \beta, \gamma \in (0, \pi)\).

The following Delambre-Gauss formulas were discovered by Delambre in 1807 (published in 1809) and were subsequently discovered independently by Gauss.


**Theorem 1.1** (Delambre-Gauss formulas for spherical triangles).

\[
\begin{align*}
\cos \frac{1}{2}(a + b) \sin \frac{1}{2} \gamma &= \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2} c, \\
\sin \frac{1}{2}(a + b) \sin \frac{1}{2} \gamma &= \cos \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2} c, \\
\cos \frac{1}{2}(a - b) \cos \frac{1}{2} \gamma &= \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2} c, \\
\sin \frac{1}{2}(a - b) \cos \frac{1}{2} \gamma &= \sin \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2} c.
\end{align*}
\]

**Remark.** Note that \(a > b\) iff \(\alpha > \beta\), and \(a + b > \pi\) iff \(\alpha + \beta > \pi\).

**Corollary 1.2** (Napier’s analogies for spherical triangles).

\[
\begin{align*}
\frac{\sin \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)} &= \tan \frac{1}{2}(a - b), \\
\frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)} &= \tan \frac{1}{2}(a + b), \\
\frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} &= \tan \frac{1}{2}(\alpha - \beta), \\
\frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} &= \tan \frac{1}{2}(\alpha + \beta).
\end{align*}
\]
Corollary 1.3 (Law of tangents for spherical triangles).
\[
\tan \frac{1}{2}(a - b) = \tan \frac{1}{2}(\alpha - \beta) \quad \tan \frac{1}{2}(a + b) = \tan \frac{1}{2}(\alpha + \beta).
\]
\[
(9)
\]
Corollary 1.4 (Law I of cosines for spherical triangles).
\[
\cos c = \cos a \cos b + \sin a \sin b \cos \gamma.
\]
\[
(10)
\]
Corollary 1.5 (Law II of cosines for spherical triangles).
\[
\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c.
\]
\[
(11)
\]
Corollary 1.6 (Law of sines for spherical triangles).
\[
\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.
\]
\[
(12)
\]
Hyperbolic triangles. Consider a triangle in the hyperbolic plane $H^2$ having side-lengths $a, b, c > 0$ and corresponding opposite interior angles $\alpha, \beta, \gamma \in (0, \pi)$.

Theorem 1.7 (Delambre-Gauss formulas for hyperbolic triangles).
\[
cosh \frac{1}{2}(a + b) \sin \frac{1}{2} \gamma = \cos \frac{1}{2}(\alpha + \beta) \cosh \frac{1}{2} c,
\]
\[
(13)
\]
\[
sinh \frac{1}{2}(a + b) \sin \frac{1}{2} \gamma = \cos \frac{1}{2}(\alpha - \beta) \sinh \frac{1}{2} c,
\]
\[
(14)
\]
\[
cosh \frac{1}{2}(a - b) \cos \frac{1}{2} \gamma = \sin \frac{1}{2}(\alpha + \beta) \cosh \frac{1}{2} c,
\]
\[
(15)
\]
\[
sinh \frac{1}{2}(a - b) \cos \frac{1}{2} \gamma = \sin \frac{1}{2}(\alpha - \beta) \sinh \frac{1}{2} c.
\]
\[
(16)
\]
Remark. Note that $a > b$ if and only if $\alpha > \beta$.

Corollary 1.8 (Law I of cosines for hyperbolic triangles).
\[
cosh c = \cosh a \cos b - \sinh a \sin b \cos \gamma.
\]
\[
(17)
\]
Corollary 1.9 (Law II of cosines for hyperbolic triangles).
\[
\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh c.
\]
\[
(18)
\]
Corollary 1.10 (Law of sines for hyperbolic triangles).
\[
\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.
\]
\[
(19)
\]
Convex right-angled hexagons in $H^2$. Consider a convex right-angled hexagon in $H^2$ having side-lengths $l_1, \ldots, l_6 > 0$ in cyclic order.

Theorem 1.11 (Delambre-Gauss formulas for convex r.a.h.’s in $H^2$).

$$\cosh \frac{1}{2}(l_1 + l_3) \sinh \frac{1}{2}l_2 = \cosh \frac{1}{2}(l_4 + l_6) \cosh \frac{1}{2}l_5, \quad (20)$$

$$\sinh \frac{1}{2}(l_1 + l_3) \sinh \frac{1}{2}l_2 = \cosh \frac{1}{2}(l_4 - l_6) \sinh \frac{1}{2}l_5, \quad (21)$$

$$\cosh \frac{1}{2}(l_1 - l_3) \cosh \frac{1}{2}l_2 = \sinh \frac{1}{2}(l_4 + l_6) \cosh \frac{1}{2}l_5, \quad (22)$$

$$\sinh \frac{1}{2}(l_1 - l_3) \cosh \frac{1}{2}l_2 = \sinh \frac{1}{2}(l_4 - l_6) \sinh \frac{1}{2}l_5. \quad (23)$$

Remark. Note that $l_1 < l_3$ if and only if $l_4 < l_6$.

Corollary 1.12 (Law of cosines for convex r.a.h.’s in $H^2$).

$$\cosh l_n = -\cosh l_{n+2} \cosh l_{n+4} + \sinh l_{n+2} \sinh l_{n+4} \cosh l_{n+3}. \quad (24)$$

Corollary 1.13 (Law of sines for convex r.a.h.’s in $H^2$).

$$\frac{\sinh l_1 \sinh l_4}{\sinh l_2} = \frac{\sinh l_3 \sinh l_6}{\sinh l_2}. \quad (25)$$

2. Trigonometric formulas for right-angled hexagons in $H^3$

Hyperbolic 3-space: $H^3$.

Right-angled hexagon in $H^3$. A r.a.h. in $H^3$ is a cyclically indexed six-tuple $(L_1, \ldots, L_6)$ of lines in $H^4$ such that, for each $n$ modulo 6, lines $L_n$ and $L_{n+1}$ intersect perpendicularly. It is said to be oriented if all the lines are oriented.

Complex (full) side-lengths $\sigma_n$ of an oriented r.a.h. in $H^3$.

For an oriented right-angled hexagon $(\vec{L}_1, \ldots, \vec{L}_6)$ in $H^3$, let $\sigma_1, \ldots, \sigma_6 \in \mathbb{C}/2\pi i \mathbb{Z}$ be respectively the complex (full) side-lengths of its side-lines $\vec{L}_1, \ldots, \vec{L}_6$.

Theorem 2.1 (Laws of cosines for oriented r.a.h.’s in $H^3$).

$$\cosh \sigma_n = \cosh \sigma_{n+2} \cosh \sigma_{n+4} + \sinh \sigma_{n+2} \sinh \sigma_{n+4} \cosh \sigma_{n+3}. \quad (26)$$

Theorem 2.2 (Laws of sines for oriented r.a.h.’s in $H^3$).

$$\frac{\sinh \sigma_1}{\sinh \sigma_2} = \frac{\sinh \sigma_3}{\sinh \sigma_4} = \frac{\sinh \sigma_5}{\sinh \sigma_6}. \quad (27)$$

Remark. The above two laws for oriented r.a.h.’s in $H^3$ were known to Schilling as early as in 1891, but a correct treatment of signs seems to be given first by Fenchel in “Elementary Geometry in Hyperbolic Space” published in 1989.
Complex half side-lengths $\delta_n$ of an oriented r.a.h. in $H^3$.

For an oriented r.a.h. $(\vec{L}_1, \cdots, \vec{L}_6)$ in $H^3$, let $\delta_n \in \mathbb{C}/2\pi i \mathbb{Z}$ be an arbitrary choice of one its two complex half side-lengths for $\vec{L}_n$, the other being $\delta_n + \pi i \in \mathbb{C}/2\pi i \mathbb{Z}$.

We obtain Delambre-Gauss formulas for oriented right-angled hexagons in $H^3$.

**Theorem 2.3 (Delambre-Gauss formulas for oriented r.a.h.’s in $H^3$).** For an oriented r.a.h. in $H^3$, there exists $\varepsilon \in \{-1, 1\}$, depending on the choices of the half side-lengths $\delta_1, \cdots, \delta_6$, so that the following formulas (28)–(31) hold:

\[
\begin{align*}
\cosh(\delta_1 + \delta_3) \cosh \delta_2 &= \varepsilon \cosh(\delta_4 + \delta_6) \cosh \delta_5, \\
-\sinh(\delta_1 + \delta_3) \cosh \delta_2 &= \varepsilon \cosh(\delta_4 - \delta_6) \sinh \delta_5, \\
-\cosh(\delta_1 - \delta_3) \sinh \delta_2 &= \varepsilon \sinh(\delta_4 + \delta_6) \cosh \delta_5, \\
\sinh(\delta_1 - \delta_3) \sinh \delta_2 &= \varepsilon \sinh(\delta_4 - \delta_6) \sinh \delta_5,
\end{align*}
\]

**Remark.** By suitably changing orientations of some of the side-lines, one may obtain the three identities (29)–(31) from the single identity (28).

3. Generalized Delambre-Gauss formulas for oriented, augmented right-angled hexagons in $H^4$

Hyperbolic 4-space: $H^4$.

Clifford algebra or the algebra of $\{e_1, e_2\}$-quaternions

$$\mathbb{A}_2 := Cl_{0,2} = \mathbb{R} + \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_1 e_2$$

subject to $e_1^2 = e_2^2 = -1$ and $e_1 e_2 + e_2 e_1 = 0$.

Reverse involution $(\cdot)^\ast : \mathbb{A}_2 \to \mathbb{A}_2$ is defined by

$$(x_0 + x_1 e_1 + x_2 e_2 + x_{12} e_1 e_2)^\ast := x_0 + x_1 e_1 + x_2 e_2 - x_{12} e_1 e_2,$$

with real coefficients $x_0, x_1, x_2, x_{12}$.

Hyperbolic functions $\cosh$ and $\sinh$ with an $\mathbb{A}_2$-variable are defined by:

$$\cosh x := \frac{\exp(x) + \exp(-x^\ast)}{2}, \quad \sinh x := \frac{\exp(x) - \exp(-x^\ast)}{2}.$$

Line and plane in $H^4$. By line and plane in $H^4$ we mean respectively complete geodesic line and totally geodesic plane in $H^4$.

Right-angled hexagon in $H^4$. A r.a.h. in $H^4$ is a cyclically indexed six-tuple $(L_1, \cdots, L_6)$ of lines in $H^4$ such that, for each $n$ modulo 6, lines $L_n$ and $L_{n+1}$ intersect perpendicularly. It is said to be oriented if all lines are oriented.
**Line-plane flag.** A line-plane flag in $H^4$ is an ordered pair $F = (L, \Pi)$, where $L$ is a line and $\Pi$ is a plane in $H^4$. It is said to be **oriented** if both the line $L$ and the plane $\Pi$ are oriented.

We say that a line $L'$ and a line-plane flag $F = (L, \Pi)$ intersect perpendicularly if $L'$ intersects each of $L$ and $\Pi$ perpendicularly.

**Augmented right-angled hexagon in $H^4.$** An a.r.a.h. in $H^4$ is a cyclically indexed six-tuple $(S_1, \cdots, S_6)$ such that either $S_1, S_3, S_5$ are all lines and $S_2, S_4, S_6$ are all line-plane flags in $H^4$, or $S_1, S_3, S_5$ are all line-plane flags and $S_2, S_4, S_6$ are all lines in $H^4$, and such that, for each $n$ modulo 6, $S_n$ and $S_{n+1}$ intersect perpendicularly. It is said to be **oriented** if all $S_n$, $n = 1, \cdots, 6$ are oriented.

**Two $e_2$-complex half distances** $\delta_F(L_1, L_2) \in (\mathbb{R} + \mathbb{R}e_2)/2\pi e_2 \mathbb{Z}$ from $L_1$ to $L_2$ along a common perpendicular $\vec{F} = (\vec{L}, \vec{\Pi})$ in $H^4$.

The two values of $\delta_F(L_1, L_2)$ differ by $\pi e_2$.

**Two $\{e_1, e_2\}$-quaternion half distances** $\delta_F(F_1, F_2) \in A_2 \mod \text{(period)}$ from $F_1$ to $F_2$ along a common perpendicular $\vec{L}$ in $H^4$.

The two values of $\delta_L(F_1, F_2)$ differ by $\pi u$ for some $u \in \sqrt{-1} \subset A_2$.

**Theorem 3.1 (Delambre-Gauss formulas for oriented a.r.a.h.’s in $H^4$).** For an oriented, augmented right-angled hexagon $(\vec{L}_1, \vec{F}_2, \vec{L}_3, \vec{F}_4, \vec{L}_5, \vec{F}_6)$ in $H^4$ with arbitrary choices of $\{e_1, e_2\}$-quaternion half side-lengths $\delta_1, \delta_3, \delta_5$ and arbitrary choices of $e_2$-complex half side-lengths $\delta_2, \delta_4, \delta_6$, the following formulas hold:

\[
\begin{align*}
\sinh \delta_1 \cosh \delta_2 \sinh \delta_3 + \cosh \delta_1 \cosh \delta_2 \cosh \delta_3 & = \varepsilon (\sinh \delta_4 \cosh \delta_5 \sinh \delta_6 + \cosh \delta_4 \cosh \delta_5 \cosh \delta_6); \\
\sinh \delta_1 \sinh \delta_2 \sinh \delta_3 - \cosh \delta_1 \sinh \delta_2 \cosh \delta_3 & = \varepsilon (\sinh \delta_4 \sinh \delta_5 \cosh \delta_6 + \cosh \delta_4 \cosh \delta_5 \sinh \delta_6); \\
\sinh \delta_1 \cosh \delta_2 \cosh \delta_3 + \cosh \delta_1 \cosh \delta_2 \sinh \delta_3 & = \varepsilon (\sinh \delta_4 \cosh \delta_5 \sinh \delta_6 - \cosh \delta_4 \cosh \delta_5 \cosh \delta_6); \\
\sinh \delta_1 \sinh \delta_2 \cosh \delta_3 - \cosh \delta_1 \sinh \delta_2 \sinh \delta_3 & = \varepsilon (\sinh \delta_4 \sinh \delta_5 \cosh \delta_6 - \cosh \delta_4 \cosh \delta_5 \sinh \delta_6),
\end{align*}
\]

with $\varepsilon = 1$ or $-1$, depending on the choices of the six half side-lengths $\{\delta_n\}_{n=1}^6$.

**Remark.** Formulas (32)–(35) above can be abbreviated as follows:

\[
\begin{align*}
(scs + ccc)_{123}^* & = \varepsilon (scs + ccc)_{456}; \\
(sss - csc)_{123}^* & = \varepsilon (ssc + ccs)_{456}; \\
(scc + ccs)_{123}^* & = \varepsilon (sss - csc)_{456}; \\
(scc - ccs)_{123}^* & = \varepsilon (ssc - css)_{456}.
\end{align*}
\]

**Remark.** The formulas (32)–(35) above are left **invariant** under taking the reverse involution ($^*$) and shifting the indices by $123456 \rightarrow 456123$. 


4. Ideas of Proof

Theorem 4.1. For an oriented r.a.h. $(\tilde{L}_1, \tilde{L}_2, \cdots, \tilde{L}_6)$ in $H^3$, let $M_n \in \text{Isom}^+(H^3)$, $n$ modulo 6, be such that $M_n(\tilde{L}_n) = \tilde{L}_n$ and $M_n(\tilde{L}_{n-1}) = \tilde{L}_{n+1}$. Then

$$M_0 M_5 M_4 M_3 M_2 M_1 = 1d.$$ (36)

Theorem 4.2. For an oriented r.a.h. $(\tilde{L}_1, \tilde{L}_2, \cdots, \tilde{L}_6)$ in $H^3$, let $M_n \in \text{Isom}^+(H^3)$, $n$ modulo 6, be as in Theorem 4.1 and let $T_n \in \text{Isom}^+(H^3)$ be a conjugate of $M_n$ such that $T_n(\tilde{L}_1) = \tilde{L}_1$ if $n = 1, 3, 5$ and $T_n(\tilde{L}_2) = \tilde{L}_2$ if $n = 2, 4, 6$. Then

$$T_1 T_2 T_3 T_4 T_5 T_6 = 1d.$$ (37)

Theorem 4.3. For an oriented a.r.a.h. $(\tilde{S}_1, \cdots, \tilde{S}_6)$ in $H^4$, let $M_n \in \text{Isom}^+(H^4)$, $n$ modulo 6, be such that $M_n(\tilde{S}_n) = \tilde{S}_n$ and $M_n(\tilde{S}_{n-1}) = \tilde{S}_{n+1}$. Then

$$M_0 M_5 M_4 M_3 M_2 M_1 = 1d.$$ (38)

Theorem 4.4. For an oriented a.r.a.h. $(\tilde{S}_1, \cdots, \tilde{S}_6)$ in $H^4$, let $M_n \in \text{Isom}^+(H^4)$, $n$ modulo 6, be as in Theorem 4.3 and let $T_n \in \text{Isom}^+(H^4)$ be a conjugate of $M_n$ such that $T_n(\tilde{S}_1) = \tilde{S}_1$ if $n = 1, 3, 5$ and $T_n(\tilde{S}_2) = \tilde{S}_2$ if $n = 2, 4, 6$. Then

$$T_1 T_2 T_3 T_4 T_5 T_6 = 1d.$$ (39)

Proof of Delambre-Gauss formulas for oriented a.r.a.h.’s in $H^4$. In the upper half-space model of $H^{n+2} \equiv \mathbb{R} + \mathbb{R}e_1 + \cdots + \mathbb{R}e_n + \mathbb{R}^+e_{n+1}$, we have

$$\text{Isom}^+(H^n) \equiv \text{PSL}(2, \Gamma_n \cup 0),$$

where $\Gamma_n \subset A_n^\mathbb{C}$ is the full Clifford group and a Vahlen matrix $A \in \text{SL}(2, \Gamma_n \cup 0)$ acts on $H^{n+2}$ as a fractional linear transformation:

$$Ax = (ax + b)(cx + d)^{-1}.$$

Note that $\Gamma_1 \cup 0 = A_1 \equiv C$ and $\Gamma_2 \cup 0 = A_2$. Now choose special positions for $\tilde{S}_1$ and $\tilde{S}_2$ as follows:

$$\tilde{S}_1 = \tilde{L}_1 = \tilde{L}_{[1, \infty]}; \quad \tilde{S}_2 = \tilde{F}_2 = (\tilde{L}_{[-1, 1]} \cdot \tilde{F}_{[-1, 1]} \cdot \varepsilon_{-1, e_1}).$$

We obtain an identity of $2 \times 2$ matrices by replacing each isometry $T_n$ in (39) by a Vahlen matrix $A_n$, and the identity isometry by $\varepsilon I$ for some $\varepsilon \in \{-1, 1\}$. Precisely, we have

$$A_1 A_2 A_3 = \varepsilon (A_4 A_5 A_6)^{-1}.$$

Working out the product matrices on both sides and equating the corresponding entries, we obtain the Delambre-Gauss formulas by suitable manipulations. □

THANK YOU!