Lecture 7. Iteration trees of transfinite length.

It is often important to continue iterating into the transfinite. The way we continue a tree $T$ of limit length $\lambda$ is: pick a branch $b$ which has been visited cofinally often below $\lambda$, and such that $M^T_b$ is well-founded. Set $M^T_{\lambda} = M^T_b$, and continue. To be more precise:

**Definition 4.** Let $\gamma \in \text{ORD}$. We call $T$ a tree order on $\gamma$ iff

1. $T$ is a strict partial order of $\gamma$;
2. $\forall \beta \gamma \gamma$ ($T$ wellorders $\forall \alpha (\alpha T \beta \gamma)$);
3. $\forall \alpha, \beta \gamma$ ($\alpha T \beta \gamma \Rightarrow \alpha \prec \beta$);
4. $\forall \alpha$ ($0 \prec \alpha \Rightarrow \alpha T \gamma$);
5. $\forall \alpha$ ($\alpha$ is a successor ordinal iff $\alpha$ is a $T$-successor);
6. $\forall \lambda \gamma$ ($\lambda$ is a limit ordinal $\Rightarrow \exists \alpha (\alpha T \gamma$ is $\epsilon$-cofinal in $\lambda$).
**Definition 2** Let \( \gamma \in \text{ORD} \). A nice iteration tree of length \( \gamma \) on \( M \)

is a system \( \mathcal{T} = \{ T, (M_\alpha, E_\alpha) \mid \alpha < \gamma \}, (i_\alpha)_{\alpha < \gamma} \) such that

1. \( T \) is a tree order on \( \gamma \),
2. \( M_0 = M \),
3. \( M_\alpha \models E_\alpha \) is a nice extender,
4. \( \alpha < \beta \Rightarrow \text{lh}(E_\alpha) < \text{lh}(E_\beta) \),
5. if \( \alpha + 1 < \gamma \), then
   \[ M_{\alpha + 1} = \text{Ult}(M_\xi, E_\alpha) \] where \( \xi \) is least s.t. \( M_\xi \prec M_\alpha \)
   and
   \[ i_\xi, \alpha + 1 = \text{canonical embedding} \],
6. if \( \lambda < \gamma \) is a limit ordinal
   \[ M_\lambda = \text{dir lim} \ M_\alpha \]
   \[ i_{\lambda \alpha} = \text{canonical embedding}, \text{ for } \alpha \in T \lambda \].
Is it always possible to continue a nice iteration tree on \( V \)? At successor steps, yes.

**Theorem 3.** Let \( M \models ZFC \) be transitive and closed under \( \omega \)-sequences. Let \( J \) be a nice iteration tree on \( M \) of length \( \alpha+1 \), and let \( \xi \leq \alpha \) be such that \( M^\xi \models M^{\alpha} \). Then \( \text{Ult}(M^\xi, E^\xi) \) is wellfounded.

**Proof.** We need the following exercise:

**Exercise 33.** Let \( J \) be a nice iteration tree \( \langle J \rangle \) and \( \alpha < \text{lh}(J) \). Show

\[
M^\alpha = \{ \langle f \rangle_\alpha \mid f \in M_0^J \land \forall \xi < \omega \exists \gamma \in [\gamma \cup \omega] \sup \{ \text{lh}(E^\gamma) \mid (\xi+1) \} \}
\]

where

\[
\gamma = \sup \{ \text{lh}(E^\xi) \mid (\xi+1) \}
\]
The exercise generalizes the fact that
\[ \text{Ult}(M, E) = \{ i^M(f)(a) \mid a \in M \wedge a \in \text{Ult}(E^\omega)^M \}. \]

It says that \( M^\alpha \) is Skolem-generated by \( \text{ran}(\nu_0^\alpha) \) together with ordinals below the sup of the lengths of extenders used on the branch 0-10-\( \alpha \).

Now let \( \xi, \lambda \) be as in the theorem, and set
\[ N = \text{Ult}(M^\xi, E^\lambda). \]

Let
\[ i : M \rightarrow N \]
be the canonical emb. (\( i^i = \pi \circ i^\alpha \))
where \( \pi : M^\alpha \rightarrow N \). Let \( \lambda = \text{Ult}(E^\lambda) \).

The proof of exercise 33 easily yields
\[ N = \{ i^M(f)(a) \mid a \in \text{Ult}(\omega)^M \}. \]

Note here that although \( N \) may be ill-founded,
\[ i(k) \in \omega \beta \beta(N) \]

where \( k = \text{crit} \left( E_{a_{\xi}} \right) \). This is because

\[ V_{i(k)+1} = V_{\lambda(k)+1} \]

and the \( \text{Ult}(M_{a_{\xi}}^E, E_{a_{\xi}}) \) can be computed in \( M_{a_{\xi}} \),

which thinks \( E_{a_{\xi}} \) is an extender.

Now pick \( \langle f_{k} | k \in \omega \rangle \) such that

there are \( a_{k} \in \text{Ult}w \) with

\[ i(f_{k+1})(a_{k+1}) \in N \quad i(f_{k})(a_{k}) \]

for all \( k \). Note \( \langle f_{k} | k \in \omega \rangle \in M \).

Thus

\[ i(\langle f_{k} | k \in \omega \rangle) = \langle i(f_{k}) | k \in \omega \rangle \in N. \]

Now pick \( \gamma \in s.t. \)

\[ N = \gamma \in \text{ORD} \land \lambda < \gamma \land \langle i(f_{k}) | k \in \omega \rangle \in V_{\gamma}. \]
Working in $N$, we have $H \subseteq \mathbb{N}$ such that

$$N \models H \text{ is transitive, } \exists \theta \in N \ s.t. \ |H| = \lambda$$

and $\pi : H \rightarrow V_\theta$ with $\pi \upharpoonright \lambda = \text{id}$ and $\langle i(f_k) | k \in \omega \rangle \in \text{ran} (\pi)$. 

Now $(\text{ran} (\pi), ^N e)$ is ill-founded in $V$, hence $(H, ^N e)$ is ill-founded in $V$.

But $H \in V^{V_N}$, $e \in \text{wfp} (N)$, a contradiction.

\[\square\]

Insofar as continuing nice trees on $V$ at limit steps goes, the main result is the following.
Theorem 4. Let \( I \) be a nice iteration tree on \( V \) of countable limit length \( \lambda \). Suppose that for all limit \( \eta < \lambda \), \( f^\mathcal{M}_\eta \) is the unique cofinal wellfounded branch of \( \mathcal{M}_\eta \). Then \( I \) has a cofinal wellfounded branch.

Remark. In other words, if \( I \) has made the only choice it could make at limit \( \eta < \lambda \), then there is a choice for it to make at \( \lambda \).

We shall sketch the proof of Theorem 4 in an appendix to this lecture. It is much like the proof of theorem 6.9.

This leads us to one of the biggest open problems in the subject.

Definition 5. nice-UBH is the statement:
Every nice iteration tree on \( V \) of limit length has at most one cofinal, wellfounded branch.
Definition 6: \textit{generic-nice-UBH} is

the statement: \[ \text{VIG} \rightarrow \text{nice-UBH} \]

whenever \( G \) is set-generic over \( V \).

Whether \textit{nice-UBH} or better \textit{generic-nice-UBH} are true are very important questions. Of course, the more useful answer would be "yes." The reason is that we would then get, via Theorem 4, an iteration strategy for \( V \). We now explain that concept more precisely.

Let \( M = \mathcal{ZFC} \) be transitive, and \( \mathcal{A} \in \text{ORD} \). The (nice) \textit{iteration} game of length \( \mathcal{A} \) on \( M \) is played as follows:

There are two players, I and II. They cooperate to produce an iteration tree \( T \) on \( M \). At successor rounds
\( \alpha + 1 \), player I extends I by picking \( E^\alpha x \) from \( M^\alpha x \), and setting \( M^\alpha x = U l \chi \left( M^\alpha x, E^\alpha x \right) \) for \( \xi \) least such that \( \text{crit}(E^\alpha x) < l h(E^\alpha x) \). (If the ultrapower is illfounded, the game ends, and I has won.) At limit rounds \( \lambda < \Theta \), II extends I by picking \( b \) cofinal in \( \lambda \) such that \( M^\alpha b \) is wellfounded. If II fails to do this, I wins.

If after \( \Theta \) rounds, I has not yet won, then II wins.

We call this game \( G_{\text{nice}}(M, \Theta) \).

A winning strategy for II in \( G_{\text{nice}}(M, \Theta) \) is called a \( \Theta \)-iteration strategy for \( M \).

We say \( M \) is \( \Theta \)-iterable iff there is such a \( \Theta \)-iteration strategy for \( M \).
We have

**Theorem 7.** If nice-UBH holds, then \( V \) is \( \omega_1 \)-iterable for nice trees.

**Proof.** Play \( \Pi \)'s strategy in \( G_{\text{nice}}(V, w_1) \) is: at round \( \lambda \), pick the unique cofinal wellfounded branch of \( T^{\lambda} \).

\( \omega_1 \)-iterability is much more useful than \( \omega_1 \)-iterability. We have

**Theorem 8.** If generic-nice-UBH holds, then \( V \) is \( K \)-iterable for nice trees, where \( K \) is the least measurable cardinal.

**Exercise 34.** Prove theorem 8. [You need to know something about preservation of large cardinals under small forcing to do]
This one, so it's really only for the more advanced students.

The main result known in the direction of proving UBH is the following. For $I$ a nice tree, set

$$s(I) = \sup \{ h(E_\alpha^{\alpha}) \mid \alpha < h(I) \}$$

$$M(I) = \bigcup_{\alpha < h(I)} \bigvee_{h(E_\alpha^{\alpha})} M_\alpha^{\alpha}.$$ 

So $\text{ORD} \cap M(I) = \bigcap_{h(I)} s(I)$, and

$M(I) = \bigvee_{h(I)} M_\alpha^{\alpha}$ for any cofinal branch

$b$ of $I$ such that $s(I) \in M_{\alpha}^{\alpha}$.
Theorem 9. Let \( \mathfrak{A} \) be a nice iteration tree of limit length \( S = S(\mathfrak{T}) \), and suppose \( b \) and \( c \) are distinct cofinal branches of \( \mathfrak{T} \). Let \( A \subseteq S \) and \( A \subseteq \mathfrak{M}^b \cap \mathfrak{M}^c \). Then

\[
(M(S), \epsilon, A) = " \mathfrak{k} (k \text{ is } A\text{-reflecting in ORD})".
\]

Remark. Another way this is often stated is:

\( S(\mathfrak{T}) \) is Woodin with respect to all \( A \subseteq \mathfrak{M}^b \cap \mathfrak{M}^c \), with respect to extenders in \( M(S) \).

Proof of Theorem 9. (Sketch.) Let us consider a special case \( \mathfrak{T} \) is an eliminating chain, and \( b \) is its even
branch, and c is its odd branch. Note that the extenders of S overlap in the following "zipper" pattern:

\[ E_n = E_n^f \]

That is, letting \( K_n = \text{crit}(E_n) \) and \( \lambda_n = \text{lh}(E_n) \):

\[ K_n < K_{n+1} < \lambda_n \]

for all \( n \). Now let \( A \subseteq S \) and \( A \in M_b \cap M_c \). Pick \( m \) large enough that

\[ A \in \text{ran}(\iota_{m_b}) \cap \text{ran}(\iota_{m+c}). \]
Claim: For any $n \geq m$

$$\iota^m_n (A \cap K_n) \cap K_n^{m+1} = A \cap K_n.$$ 

(What is $\iota^m_n$ shifts $A$ to itself below the next critical points. It doesn't matter whether we write $\iota^m_n$ or $\iota^{M_m^2}_{n+1}$ here, since the ultrapowers are the same below the image of $K_n$.)

**Proof:** Suppose e.g. $a + 1 \in b$. Let

$$A = \iota_{m+1}^n (\overline{A}).$$

Then $\iota_{m+1}^n (\overline{A}) \cap K_n = A \cap K_n$, because $\text{crit}(\iota_{m+1}^n) = K_n$.

So $\iota_{m+1}^n (A \cap K_n)$ agrees with $A$ below $\iota_{m+1}^n (K_n)$. But $\iota_{m+1}^n (A \cap K_n)$ agrees with $\iota^m_n (A \cap K_n)$ below $K_n$, because $\text{crit}(\iota_{m+1}^n) \geq \lambda_n$. 


Consider the embedding $\mathcal{L} \subseteq \text{End}_{representing}$.

\[
\mathcal{L} \subseteq \mathcal{L}_{\text{End}_{representing}} \subseteq \mathcal{L}_{\text{End}_{K_m+1}} = \mathcal{L}
\]

Note: $E_i \subseteq K_{i+1} \subseteq M(I)$, because $K_{i+1} \subseteq \text{lh}(E_i)$. It is routine to show that $\mathcal{L}$ witnesses $K_m$ is $A$-reflecting and $\beta$ in $M(I)$.

Exercise 35 (a) Complete the proof of this.

(b) Complete the proof of Theorem 9 as follows! Let $b$ and $c$ be distinct cotinal branches of $\mathcal{L}$. Find $\langle \alpha_n \text{ new} \rangle$ cotinal in $\mathcal{L}$ such that $b \neq c$. (Hint?)
\[
\alpha_{2n} + 1 \in b, \\
\alpha_{2n+1} + 1 \in c,
\]

and

\[
\text{crit}(E_{\alpha_{2n}}) < \text{crit}(E_{\alpha_{2n+1}}) < \text{lh}(E_{\alpha_{2n}})
\]

for all \( k \), i.e. we have the zipper pattern embedded in the two branches. Now argue as above.

Remark: For more detail, see

"Iteration trees" (Martin, Steel)

JAMS.

Remark: So if \( J \) has distinct cofinal branches, then \( \text{lh}(J) \) has cofinality \( \omega \).

This is also easy to see from the fact that every branch of an iteration tree is closed below its sup (as a set of ordinals).
Corollary 10 Suppose nice-UBH fails. Then there is a proper class model with a Woodin cardinal.

Proof Let $\mathcal{A}$ be nice on $V_\xi$ and have distinct cofinal wellfounded branches $b$ and $c$. Then

\[ L(CM_b) \subseteq L(CM_c) \subseteq M_b \cap M_c. \]

So by Proposition 9,

\[ L(CM(\xi)) \models \text{S(\xi)} \text{ is Woodin}. \]