Constructions of Optimal Variable-Weight OOCs

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Outline

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1 Introduction

1.1. Background

Optical orthogonal codes (OOCs) were introduced by Salehi, as signature sequences to facilitate multiple access in optical fibre networks [1-2].

OOCs have found wide range of applications such as mobile radio, frequency-hopping spread-spectrum communications, radar, sonar, collision channel without feedback and neuromorphic network [3-7].


The following are from [1]. In FO-CDMA, there are $N$ transmitter and receiver pairs (users). Fig. 1 shows one such network. The set of FO-CDMA optical pulse sequences essentially becomes a set of address codes or signature sequences for the network. To send information from user $a$ to user $b$, the address code for receiver $b$ is impressed upon the data by the encoder at the $j$th node. One of the primary goals of FO-CDMA is to extract data with the desired optical pulse sequence in the presence of all other users’ optical pulse sequences. So, the sequences should be designed to satisfy the following two conditions, namely:

(C1) Each sequence can easily be distinguished from a shifted version of itself (auto-correlation property);
(C2) each sequence can be easily distinguished from (a possibly shifted version of) every other sequence in the set (cross-correlation property).
Fig. 1 Code-division multiple-access optical system.
\((v, k, \lambda_a, \lambda_c)\)-OOC \(\mathcal{C}\) A family of \((0, 1)\) sequences of length \(v\) and weight \(k\) satisfying the following two properties:

- **Auto-correlation:** For any \(x = (x_0, x_1, \ldots, x_{v-1}) \in \mathcal{C}\), and any integer \(\tau, 0 < \tau < v\),
  \[
  \sum_{t=0}^{v-1} x_t x_{t+\tau} \leq \lambda_a,
  \]
  where the summation is carried out by treating binary symbols as reals.

- **Cross-correlation:** Similarly, for any \(x \neq y, x = (x_0, x_1, \ldots, x_{v-1}) \in \mathcal{C}\), \(y = (y_0, y_1, \ldots, y_{v-1}) \in \mathcal{C}\), and any integer \(\tau\),
  \[
  \sum_{t=0}^{v-1} x_t y_{t+\tau} \leq \lambda_c.
  \]
\((v, k, \lambda)\)-OOC \(\mathcal{C}\) A \((v, k, \lambda_a, \lambda_c)\)-OOC with property that \(\lambda_a, \lambda_c = \lambda\).

**Example 1.1** A \((37, 4, 1)\)-OOC with 3 codewords.

110100000000000000000000100000000000000
10010000100000100000000000000000000000000
1000001000000001000000100000000000000000000.

Most existing work on OOC’s have assumed that all codewords have the same weight. In general, the variable-weight OOCs can generate larger code size than that of constant-weight OOCs [8].

In 1996, G-C Yang introduced multimedia optical CDMA communication system employing variable-weight OOCs [10]. The variable-weight property of the OOCs enables the system to meet multiple QoS requirement. Recently variable-weight OOCs have attracted much attention [8-10].

Let $W = \{w_0, w_1, \ldots, w_p\}$, $L = \{\lambda^0_a, \lambda^1_a, \ldots, \lambda^p_a\}$ and $Q = \{q_0, q_1, \ldots, q_p\}$

**Definition:** An $(n, W, L, \lambda_c, Q)$ variable-weight optical orthogonal code $C$, or $(n, W, L, \lambda_c, Q)$-OOC, is a collection of binary $n$-tuples such that the following three properties hold:

- **Weight Distribution:** Every $n$-tuple in $C$ has a Hamming weight contained in the set $W$; furthermore, there are exactly $q_i|C|$ codewords of weight $w_i$, i.e., $q_i$ indicates the fraction of codewords of weight $w_i$.

- **Periodic Auto-correlation:** For any $x = (x_0, x_1, \ldots, x_{n-1}) \in C$ with Hamming weight $w_i \in W$, and any integer $\tau$, $0 < \tau < n$,

$$\sum_{t=0}^{n-1} x_t x_{t\oplus\tau} \leq \lambda^i_a,$$

where the summation is carried out by treating binary symbols as reals.

- **Periodic Cross-correlation:** Similarly, for any $x \neq y$, $x = (x_0, x_1, \ldots, x_{n-1}) \in C$, $y = (y_0, y_1, \ldots, y_{n-1}) \in C$, and any integer $\tau$,

$$\sum_{t=0}^{n-1} x_t y_{t\oplus\tau} \leq \lambda_c.$$
Example 1.2  A (80, {4, 5}, 1, {1/2, 1/2})-OOC with 4 codewords.

1001000000000000000000001000000000000000000000100000
000010000000000000000000000000000000000000000000000,
10010000000000001010000000000000000000000000000000000
000000000000001000000000000000000000000000000000000,
110000010000000000000000000000000010000000000000000
000000000000000000000000000000000000000000000000000,
1000000000000000010000000100000000000000000000000000
000000000000000000000000000000000000000000000000000.


We give an example to show that variable-weight OOCs can generate larger code size than that of constant-weight OOCs. Let
\[ C^{(1)} = \{110100010000000010000000000000000\}; \]
\[ C^{(2)} = \{11010000100000000000000010000000000000000, \]
\[ 100000000001000000000000000000000010000000000000000, \]
\[ 10000000000000000000000000100010000000000\}; \]
\[ C^{(3)} = \{110100010000000010000000000000000, \]
\[ 1000010000000000000100000000000000000000, \]
\[ 10000000100000000000000000000010000000000, \]
\[ 100000000000000001000000000001000000000000\}. \]

Then \( C^{(1)} \) is an optimal \((40, 5, 1)\)-OOC (only one codeword), \( C^{(2)} \) is an optimal \((40, \{3, 4, 5\}, 1, \{1/3, 1/3, 1/3\})\)-OOC (three codewords), \( C^{(3)} \) is an optimal \((40, \{3, 5\}, 1, \{3/4, 1/4\})\)-OOC (four codewords).
1.2. New Upper Bound

An upper bound on the size of variable-weight OOCs was given in [9].

Let $\Phi(n, W, L, \lambda, Q)=\max\{|C| : C \text{ is an } (n, W, L, \lambda, Q)-\text{OOC}\}$.

Lemma 1.1 Let $\lambda^i_a \geq \lambda (0 \leq i \leq p)$. Then

$$\Phi(n, W, L, \lambda, Q) \leq N, \quad N = \left\lfloor \frac{(n-1)(n-2)\ldots(n-\lambda)}{\sum_{i=0}^{p} q_i w_i (w_i-1)(w_i-2)\ldots(w_i-\lambda)/\lambda^i_a} \right\rfloor.$$

The bound is not tight some times. The following is an example. Let $n = 64$, $W = \{3, 4\}$, $L = \{1, 1\}$, $\lambda = 1$, $Q = \{1/2, 1/2\}$, then $\left\lfloor \frac{v-1}{\sum_{i=0}^{1} q_i w_i (w_i-1)} \right\rfloor = 7$. It is clear that there does not exist a $(64, \{3, 4\}, \{1, 1\}, 1, \{1/2, 1/2\})$-OOC with 7 codewords since the number of codewords of weight 3 and 4 are equal.
\( q_i = b_i/a_i, \ \gcd(a_i, b_i) = 1, \ 0 \leq i \leq p. \)

\( f(Q) = \text{lcm}(a_0, a_1, \ldots, a_p), \ f_i(Q) = f(Q)q_i, \) then \( q_i = f_i(Q)/f(Q). \)

**Theorem A1 (New upper bound)** Let \( \lambda^i_a \geq \lambda (0 \leq i \leq p). \) Then

\[ \Phi(n, W, L, \lambda, Q) \leq f(Q)\left\lfloor \frac{N}{f(Q)} \right\rfloor. \]

**Proof** Suppose \( \mathcal{C} \) is a \((n, W, L, \lambda, Q)\)-OOC with \(|\mathcal{C}| = \Phi(n, W, L, \lambda, Q)\). The number of codewords of weight \( w_i \) is \( \Phi(n, W, L, \lambda, Q)q_i = \Phi(n, W, L, \lambda, Q)b_i/a_i, \ 0 \leq i \leq p. \) \( \gcd(a_i, b_i) = 1, \) then \( a_i|\Phi(n, W, L, \lambda, Q), \) and hence \( f(Q)|\Phi(n, W, L, \lambda, Q). \)

A \((n, W, L, \lambda, Q)\)-OOC \( \mathcal{C} \) is **optimal** if \( \Phi(n, W, L, \lambda, Q) \) meets the above upper bound.
1.3. **Known Results**

\((n, W, \lambda, Q)\)-OOC: an \((n, W, L, \lambda_c, Q)\)-OOC with the property that 
\[ \lambda_0^a = \lambda_1^a = \ldots = \lambda_p^a = \lambda_c = \lambda. \]

Yang [26]: \((n, \{s, s + 1\}, \{1, 1\}, 1, Q)\)-OOCs, \((n, \{s, s + 1\}, \{2, 2\}, 1, Q)\)-OOCs, \((n, \{2s, s\}, \{2, 1\}, 1, Q)\)-OOCs, \((n, \{2s, s\}, \{2, 2\}, 1, Q)\)-OOCs. Some of them are optimal, and \(|W| = 2\).

See Gu and Wu [25], I-B Djordjevic [27] also.
2 Cyclic Packings and Optimal OOCs

2.1. Optimal $2$-CP($W, 1, Q; v$)
s
$B \subseteq Z_v$, $\Delta B = \{x - y : x, y \in B, x \neq y\}$.

$2$-CP($W, 1; v$) $\mathcal{F}$ $\mathcal{F}$ is a set of subsets of $Z_v$ such that $\Delta \mathcal{F} = \bigcup_{B \in \mathcal{F}} \Delta B$ covers each nonzero element of $Z_v$ at most once. For each $B \in \mathcal{F}$, $|\{B + g : g \in Z_v\}| = v$.

$W = \{|B| : B \in \mathcal{F}\}$.

Note that a $2$-CP($W, 1, v$) is a CP($W, 1; v$) in [11].

$2$-CP($W, 1, Q : v$) $\mathcal{F}$ A $2$-CP($W, 1 : v$) with the property that the fraction of blocks of size $w_i$ is $q_i$, $0 \leq i \leq p$.

Remark Cyclic ($v, W, 1$)-DF: $\Delta \mathcal{F}$ covers each nonzero element of $Z_v$ exactly once.

$CD_1(W, Q, 2; v)$ The maximum number of base blocks in any $2$-CP($W, \lambda, Q; v$).

Lemma 2.1 \[ CD_1(W, Q, 2; v) \leq f(Q)\left\lfloor \frac{N}{f(Q)} \right\rfloor. \]

A 2-CP\((W, 1, Q : v)\) is optimal if \( CD_1(W, Q, 2; v) \) meets the above upper bound.

\( \mathcal{F} \) is a 2-CP\((W, 1, v)\). DL\((\mathcal{F}) = Z_v \setminus \Delta\mathcal{F}. \)

\( g \)-regular 2-CP\((W, 1, v)\) \( \mathcal{F} \) \( \text{If DL}(\mathcal{F}) \) forms a group of order \( g \), then the 2-CP\((W, 1, v)\) \( \mathcal{F} \) is called \( g \)-regular.

Motivation of \( g \)-regular: For recursive construction

Lemma 2.2 \( \text{If } 1 \leq g \leq w, \text{ then the } g \)-regular 2-CP\((W, 1, v; Q)\) is optimal, where \( w = \sum_{i=0}^{p} f_i(Q)w_i(w_i - 1). \)
Example 2.1  An optimal 2-CP(\{3, 4\}, 1, \{1/2, 1/2\}; 64) \mathcal{B} with 6 base blocks.

\[ \mathcal{B} = \{\{0, 1, 3, 7\}, \{0, 5, 13, 22\}, \{0, 10, 21, 33\}, \{0, 14, 29\}, \{0, 16, 34\}, \{0, 19, 39\}\}. \] Then \mathcal{B} forms a 2-CP(\{3, 4\}, 1, \{1/2, 1/2\}; 64).

\[ CD_1(\{3, 4\}, \{1/2, 1/2\}, 2; 64) \leq 6. \] So, the packing is optimal.
Given an optimal 2-CP($W, 1, Q; v$), one can construct a $(0, 1)$-sequences $\mathcal{C}$ of length $v$, and weight $w_i$ from a base block of size $w_i$ whose nonzero bit positions are exactly indexed by the base block. It is not difficult to see that the derived $(0, 1)$-sequences form a cyclic $(v, W, 1, Q)$-OOC.

Conversely, suppose $\mathcal{C}$ is an optimal $(v, W, 1, Q)$-OOC. For each codeword, we construct a $w_i$-subset of $\mathbb{Z}_v$ by taking the index set of its nonzero bit positions, where $w_i \in W$. This creates a family $\mathcal{F}$ of subsets of $\mathbb{Z}_v$. From the correlation properties of the OOC, $\mathcal{F}$ forms a 2-CP($W, 1, Q; v$). Therefore, we have the following result [12].

**Lemma 2.3** An optimal 2-CP($W, 1, Q; v$) is equivalent to an optimal $(v, W, 1, Q)$-OOC.

2.2. Recursive Constructions

\((m, k; 1)-CDM\) A \(k \times m\) matrix \(D = (d_{ij})\) \((0 \leq i \leq k - 1, 0 \leq j \leq m - 1)\) whose each entry is an integer of \(Z_m\) such that for any two distinct rows \(i_1\) and \(i_2\), the list of \(d_{i_1j} - d_{i_2j}\) \((j = 0, 1, \ldots m - 1)\) contains each integer of \(Z_m\) exactly once. It is easy to see that if an \((m, k; 1)\)-CDM exists, then so does an \((m, h; 1)\)-CDM for any positive integer \(h \leq k\).

**Example 2.3** A \((5, 5; 1)\)-CDM

\[
\begin{array}{ccccc}
00000 \\
01234 \\
02413 \\
03142 \\
04321 \\
\end{array}
\]
When \( m \geq k \) is an odd prime, an \((m, k; 1)\)-CDM can be constructed by simply taking \( d_{ij} = ij \mod m \) for \( 0 \leq i \leq k - 1 \) and \( 0 \leq j \leq m - 1 \).

**Lemma 2.4** If \( m \geq k \) is an odd prime, then there exists an \((m, k; 1)\)-CDM.

**Lemma 2.5** ([13]) If \( m \geq 5 \) is odd, and \( \gcd(m, 27) \neq 9 \), then there exists an \((m, 4; 1)\)-CDM.

Lemma 2.6 Suppose that both a $g$-regular $2$-CP($W, 1, Q; v$) and an optimal $2$-CP($W, 1, Q; g$) exist, then an optimal $2$-CP($W, 1, Q; v$) exists. Moreover, if the given $2$-CP($W, 1, Q; g$) is $r$-regular, then so is the derived $2$-CP($W, 1, Q; v$).

Lemma 2.7 Suppose that there exist:
(1) a $g$-regular $2$-CP($W, 1, Q; v$);
(2) an $(m, w_p; 1)$-CDM;
(3) an optimal $2$-CP($W, 1, Q; gm$).
Then there exist both a $gm$-regular $2$-CP($W, 1, Q; mv$) and an optimal $2$-CP($W, 1, Q; mv$). Moreover, if the given $2$-CP($W, 1, Q; gm$) is $r$-regular, then so is the derived $2$-CP($W, 1, Q; mv$).

Lemma 2.8 Suppose that $q_1$ and $q_2$ are two odd primes, and $q_2 \geq w_p$. If both a $g$-regular $2$-CP($W, 1, Q; gq_1$) and an optimal $2$-CP($W, 1, Q; gq_2$) exist, then so does an optimal $2$-CP($W, 1, Q; gq_1q_2$). Moreover, if the given $2$-CP($W, 1, Q; gq_2$) is $g$-regular, then so is the derived $2$-CP($W, 1, Q; gq_1q_2$).
2.3. **Optimal** $(v, W, 1, \{1/2, 1/2\})$-OOCs

**Theorem B1** If $u \equiv 9 \pmod{18} > 9$ is an integer, then there exists a $(u, \{3, 4\}, 1, \{1/2, 1/2\})$-OOC.

We need only to prove that if $u \equiv 9 \pmod{18} > 9$ is an integer, then there exists a 9-regular and an optimal 2-CP$(\{3, 4\}, 1, \{1/2, 1/2\}; u)$ (The packing is clear optimal).

We will use the following notations in the sequel. For $B = \{(a_1, i_1), (a_2, i_2), \ldots, (a_k, i_k)\} \in (\mathbb{Z}_q \times \mathbb{Z}_m)^k$, $\Delta B = \{(a_{j_2} - a_{j_1}, i_{j_2} - i_{j_1}) : 1 \leq j_1 < j_2 \leq k\}.$ let

$$D_s = \{d : (d, s) \in \Delta B\}, s \in \mathbb{Z}_m.$$
Skew starter $G$ is an Abelian group of order $v$, $S = \{\{s_i, t_i\} : 1 \leq i \leq \frac{v-1}{2}\}$ satisfies the following properties:

1. $\{s_i : 1 \leq i \leq \frac{v-1}{2}\} \cup \{t_i : 1 \leq i \leq \frac{v-1}{2}\} = G \setminus \{0\}$;
2. $\{\pm(s_i - t_i) : 1 \leq i \leq \frac{v-1}{2}\} = G \setminus \{0\}$.
3. $\{\pm(s_i + t_i) : 1 \leq i \leq \frac{v-1}{2}\} = G \setminus \{0\}$.

As stated in [14], let $X = \{S_i : 1 \leq i \leq (v - 1)/2\}$, $Y = \{t_i : 1 \leq i \leq (v - 1)/2\}$, then we may assume that $X = -Y$ and hence we have $X \cup (-X) = Y \cup (-Y) = X \cup Y = G \setminus \{0\}$.

**Example 2.4** A skew starters $S_v$ in $Z_v$, $v = 7, 11$

$S_7 = \{\{1, 5\}, \{2, 3\}, \{4, 6\}\}$,
$S_{11} = \{\{1, 2\}, \{3, 6\}, \{4, 8\}, \{5, 10\}, \{9, 7\}\}$.

Skew starters were used to construct generalized Steiner systems and optimal constant-weight optical orthogonal codes (OOCs), see [14-16]. For more details on skew starters, the interested reader may refer to [17-18]. The following result was stated in [15].

**Lemma 2.9** There exists a skew starter in $Z_v$ for each positive integer $v$ such that $\gcd(v, 6) = 1$, $v$ is not divisible by 5 or is divisible by 25. There does not exist any skew starter in $Z_v$ if $v \equiv 0 \pmod{3}$.

From Lemma 2.9, it is clear that there exists a skew starter in $Z_v$ if $\gcd(v, 30) = 1$.

Sketch for Proof of Theorem B1

Step 1 If there exists a skew starter in $Z_q$, then there exists a $r$-regular 2-CP($\{3, 4\}, 1, \{1/2, 1/2\}; 9q$), $r \in \{9, 27\}$. The packing is optimal when $r = 9$.

We only prove the case of $r = 9$. Since a skew starter exists in $Z_q$, then $\gcd(q, 3) = 1$, and hence $Z_q \times Z_9$ is isomorphic to $Z_{9q}$.

Suppose that $S = \{\{x_i, y_i\} : 1 \leq i \leq t\}$ is a skew starter in $Z_q$, where $t = (q - 1)/2$. Define the base blocks as follows.

$$A_1^i = \{(x_i, 0), (y_i, 0), (0, 2), (x_i + y_i, 3)\},$$
$$A_2^i = \{(x_i + y_i, 0), (0, 1), (x_i, 5)\}, 1 \leq i \leq t.$$

We compute the differences from them. It is easy to see that $D_s = -D_{9-s}$, $5 \leq s \leq 8$. So, we need only to consider $D_j$ for $0 \leq j \leq 4$.

$$D_0 = \bigcup_{i=1}^{t} \{(x_i - y_i), -(x_i - y_i)\},$$
$$D_1 = \bigcup_{i=1}^{t} \{(x_i + y_i), -(x_i + y_i)\},$$
$$D_2 = \bigcup_{i=1}^{t} \{-x_i, -y_i\}, D_3 = D_4 = \bigcup_{i=1}^{t} \{x_i, y_i\}.$$
Now set $\mathcal{F} = \{ A_i^1 : \leq i \leq t \} \cup \{ A_i^2 : \leq i \leq t \}$, then $\Delta \mathcal{F}$ does cover each element of $(\mathbb{Z}_q \times \mathbb{Z}_9) \setminus (\{0\} \times \mathbb{Z}_9)$ exactly once, while any element of the additive subgroup $\{0\} \times \mathbb{Z}_9$ is not covered at all. So, $(\mathbb{Z}_9, \mathcal{F})$ forms a 9-regular and optimal 2-CP($\{3, 4\}, 1, \{1/2, 1/2\}; 9q$).

Since $w = 1 \times 2 \times 3 + 1 \times 4 \times 3 = 18$ and $g = 9 < w$, then the packing is optimal from Lemma 2.3. This completes the proof.
The case of \( u \equiv 9, 45 \pmod{54} \)

**Step 2** There exist a 9-regular 2-CP(\( \{3, 4\}, 1, \{1/2, 1/2\}; 9 \times 5^i \), \( i \geq 1 \) is an integer.

\( \mathcal{F}(1) = \{\{0, 4, 12, 26\}, \{0, 1, 7, 28\}, \{0, 2, 11\}, \{0, 3, 16\}\}. \)

Then \( \mathcal{F}(1) \) forms a 9-regular 2-CP(\( \{3, 4\}, 1, \{1/2, 1/2\}; 45 \)). By recursive construction, one can prove the result.
Step 3 Write $u = 9 \times 5^i v$, $i$, $j$ are nonnegative integers, and $5 \nmid v$. Then $v$ is odd and $(v, 30) = 1$.

If $v = 1$, then $i \geq 1$, the conclusion comes from Step 2.

If $v > 1$, then a skew starter in $Z_v$ exists, and hence a 9-regular and an optimal 2-CP($\{3, 4\}, 1, \{1/2, 1/2\}; 9v$) exist from Step 1. If $i = 0$, then the conclusion is true since a 9-regular and an optimal 2-CP($\{3, 4\}, 1, \{1/2, 1/2\}; 9v$) exist. If $i \geq 1$, then the conclusion comes the existence of a 9-regular 2-CP($\{3, 4\}, 1, \{1/2, 1/2\}; 9v$) from Step 1, a 9-regular 2-CP($\{3, 4\}, 1, \{1/2, 1/2\}, 9 \times 5^i$) from Step 2, the $(v, 4 : 1)$-CDM from Lemma 2.6, and Lemma 2.9.
The case of \( u \equiv 27 \pmod{54} \)

**Step 4** There exists a 27-regular 2-CP(\{3, 4\}, 1, \{1/2, 1/2\}; 27 \times 5^i) for \( i \geq 1 \). There exists a 9-regular 2-CP(\{3, 4\}, 1, \{1/2, 1/2\}; 9 \times 3^i) for \( i > 11 \). There exists an optimal 2-CP(\{3, 4\}, 1, \{1/2, 1/2\}; 27).

Direct construction: 27-regular 2-CP(\{3, 4\}, 1, \{1/2, 1/2\}; 27 \times 5); 9-regular 2-CP(\{3, 4\}, 1, \{1/2, 1/2\}; 9 \times 3^i), \( i = 2, 3, 4 \).
Step 5  If $v = 27s$, $v > 27$, and $\gcd(6, s) = 1$, then there exists a 27-regular 2-CP($\{3, 4\}, 1, \{1/2, 1/2\}; v$).

Proof  Write $s = 5^i s_1$. If $s_1 = 1$, then $i \geq 1$, the conclusion comes from Step 4. If $s_1 > 1$, then $\gcd(s_1, 30) = 1$, a skew starter in $\mathbb{Z}_{s_1}$ exists. If $i = 0$, a 27-regular 2-CP($\{3, 4\}, 1, \{1/2, 1/2\}; 27 \times s_1$) exists from Step 1. If $i > 0$, for $0 \leq j \leq i - 1$, applying Lemma 2.7 with $g = 27$, $v = 27 \times s_1 \times 5^j$, $m = 5$, one can obtain a 27-regular 2-CP($\{3, 4\}, 1, \{1/2, 1/2\}; v$).
The Last Step  Write $u = 54t + 27 = 27(2t + 1) = 27 \times 3^i v$, $\gcd(6, v) = 1$. If $v = 1$, then $u = 27 \times 3^i$. If $i = 0$, then an optimal $(27, \{3, 4\}, 1, \{1/2, 1/2\})$-OOC exists from Step 4. If $i \geq 1$, then $u = 9 \times 3^{i+1}$, and $i + 1 \geq 2$, an optimal $(u, \{3, 4\}, 1, \{1/2, 1/2\})$-OOC exists from Step 4.

For $v > 1$, since $\gcd(6, v) = 1$, then $9v \equiv 9, 45 \pmod{54}$. If $i = 0$, then the result comes from Step 5. For $i \geq 1$, $u = 9 \times 3^{i+1} v$, $i + 1 \geq 2$. A 9-regular 2-CP$(\{3, 4\}, 1, \{1/2, 1/2\}; 9 \times 3^{i+1})$ exists from Step 4, a 9-regular 2-CP$(\{3, 4\}, 1, \{1/2, 1/2\}; 9v)$ exists from Step 1, a $(3^{i+2}, 4; 1)$-CDM exists from Lemma 2.5. So, there exists a 9-regular 2-CP$(\{3, 4\}, 1, \{1/2, 1/2\}; u)$ from Lemma 2.7, the packing is also optimal. This completes the proof.
Similarly, we can obtain the following results.

**Theorem B2** If \( u \equiv 16, 80 \pmod{96} \geq 16 \) is an integer, then there exists an optimal \((u, \{4, 5\}, 1, \{1/2, 1/2\})\)-OOC.

**Theorem B3** There exists an optimal \((9v, \{3, 4\}, 1, \{4/5, 1/5\})\)-OOC for any integer \( v \) whose prime factors are all congruent to 1 modulo 4 and no less than 5.

**Theorem B4** If \( u \equiv 15, 75 \pmod{90} \geq 16 \) is an integer, then there exists an optimal \((u, \{3, 4\}, 1, \{1/3, 2/3\})\)-OOC.

**Theorem B5** If \( u \equiv 13 \pmod{26} \geq 13 \) is an integer, then there exists an optimal \((u, \{3, 5\}, 1, \{1/2, 1/2\})\)-OOC.

[20] H. Zhao, D. Wu and Z. Mo, Further results on optimal \((v, \{3, k\}, 1, \{1/2, 1/2\})\)-OOCs for \(k = 4, 5\), submitted to DM.
3 Problems for Further Research

To increase the codewords of constant weight OOCs, Yang gave an upper bound on codeword size of \((v, k, 2, 1)\)-OOCs, and presented constructions of \((v, k, 2, 1)\)-OOCs. Unfortunately, the bound does not tight in many cases. Recently, Momihara and Buratti presented a tight upper bound on the codeword size of a \((v, k, 2, 1)\)-OOC for \(k = 3, 4\), and many new optimal \((v, 4, 2, 1)\)-OOCs were also constructed. So, the following problems are worth to studying.

**Problem 1** Construct more optimal \((v, 4, 2, 1)\)-OOCs.

**Problem 2** Find tight upper bound on the codeword size of a \((v, k, 2, 1)\)-OOC for \(k \geq 5\), and construct some optimal \((v, 5, 2, 1)\)-OOCs.
Although many new optimal variable-weight OOCs had been constructed in this research report, further optimal constructions are also needed. For \((v, \{3, 4\}, 1, Q)\)-OOCs, it is proved that there exists an optimal \((v, \{3, 4\}, 1, \{1/2, 1/2\})\)-OOC for any integer \(v \equiv 9, 45 \pmod{54}\) (Theorem 3.3.13). Recently, Zhao, Wu and Fan proved that there exists an optimal \((v, \{3, 4\}, 1, \{1/2, 1/2\})\)-OOC for any integer \(v \equiv 9 \pmod{18}\). For \(v \equiv 0 \pmod{18}\), Zhao, Wu and Fan proved that there exists an optimal \((v, \{3, 4\}, 1, \{1/2, 1/2\})\)-OOC for any integer \(v \equiv 18, 90 \pmod{108}\).

In this report, it is also proved that there exists an optimal \((v, \{4, 5\}, 1, \{1/2, 1/2\})\)-OOC for any integer \(v \equiv 16, 80 \pmod{96}\) (Theorem 3.3.14). There exists an optimal \((v, \{3, 4\}, 1, \{1/3, 2/3\})\)-OOC for any integer \(v \equiv 15, 75 \pmod{90}\) (Theorem 3.3.16). So, the following problems are nature.

**Problem 3** Construct more congruent classes of optimal \((v, \{3, 4\}, 1, \{1/2, 1/2\})\)-OOCs, for example \(v \equiv m \pmod{18}, m \not\in \{0, 9\} \).
Problem 4 Construct congruent classes of optimal \((v, \{3, 4\}, 1, Q)\)-OOCs for \(Q \not\in \{\{1/2, 1/2\}, \{1/3, 2/3\}\}\).

Problem 5 Construct congruent classes of optimal \((v, W, 1, \{1/2, 1/2\})\)-OOCs for \(W \neq \{3, 4\}\), for examples \(W \in \{\{3, 5\}, \{4, 5\}\}\). \{3, 6\}\).

Very little infinite classes of optimal \((v, W, 1, Q)\)-OOCs is known for \(|W| \geq 3\). So, the following problem is need to be studied.

Problem 6 Construct infinite classes of optimal \((v, W, 1, Q)\)-OOCs for \(|W| \geq 3\), for examples \(W \in \{\{3, 4, 5\}, \{3, 4, 6\}\}\).
Thank You!