The Theory of Evolutionary Conflicts

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Singapore, May 2011.
The **biological** questions

(1) Why do some individuals in a population behave aggressively and others timidly in **intra-specific** conflicts?

(2) Why do individuals equipped with substantial weapons (usually) not use them in **intra-specific** conflicts?

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**NOT INTER-SPECIFIC** e.g. **NOT PREDATOR PREY**
Above courtesy of
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Price, and Maynard Smith and Price (1974) introduced the notion of an Evolutionary Stable Strategy (ESS) to attempt to answer the biological issue. **An ESS is strategy which if played by a population is stable, in the sense that no other strategy can invade.**
The **mathematical** questions

(1) In a population where there are various possible strategies available (and specific rules of interaction, with payoffs and cost) which strategies persist, and with what frequencies?

(2) Can we incorporate (a) multi-player interactions, (b) multiple reward levels, (c) random immigrations etc etc.
1. The Basic Notion of an Evolutionary Conflict.
2. Evolutionary Stable Strategies (symmetric, 2 players).
4. Replicator Dynamics, continuous and discrete time.
5. Dynamics of three strategy cyclic systems, War of Attrition.
Pairwise, symmetric conflicts. Set of possible plays $M$. If an individual in a contest (a one-on-one competition) plays $i \in M$ against an individual who plays $j \in M$ then that player receives a reward of $a(i, j)$. Note this includes any costs incurred and may be negative.
Now we assume we have an infinite, randomly mixed population; this allows us to work with expected payoffs. We say the population is playing strategy \( p \), if \( p \) is the probability (density) over \( M \) of the strategies in that population. For ease I will write as if \( M \) were discrete.
Payoffs are additive. If an individual is playing a $i$ in a population with strategy $p$ (where the probability of playing $j$ is $p_j$) then its expected payoff is just $\sum_j a(i, j)p_j$ which we write as $E(i, p)$. If an individual is playing a mixed strategy $q$ (i.e. picking strategy $i$ with probability $q_i$) then its expected payoff is $\sum_i q_i E(i, p)$, which we write as $E(q, p)$. This is also the expected payoff to an individual picked at random from a sub-population playing $q$ in a population playing $p$. For consistency we now write $a(i, j) = E(i, j)$. 
Population is stable wrt invasions
Suppose a population is playing \( p \), then the payoff to that population is \( E(p, p) \). Suppose there is an individual who plays \( q \) (or a group of players playing \( q \)). If a small group (frequency \( \epsilon \)) of such individuals is present in the population then they will have payoffs per individual \( E(q, (1 - \epsilon)p + \epsilon q) \) while the population as a whole will have payoff per individual \( E(p, (1 - \epsilon)p + \epsilon q) \). Thus \( q \) will not invade provided \( E(q, (1 - \epsilon)p + \epsilon q) < E(p, (1 - \epsilon)p + \epsilon q) \).
We say that $p$ is Evolutionarily Stable (ES) wrt $q$ if

(1) $E(q, p) < E(p, p)$,

or

(2) $E(q, p) = E(p, p)$ and $E(q, q) < E(p, q)$
Then we say that $p$ is an ESS if $p$ is ES wrt every $q \neq p$. 
Maynard Smith and Price definition of an ESS.

\( p \) is an ESS if, for every \( q \neq p \), \( E(p, p) \geq E(q, p) \), and if \( E(p, p) > E(q, p) \) then \( E(p, q) > E(q, q) \).
The Hawk-Dove conflict.

There is a reward $V$ available. There are strategies $D=$Dove (always retreat) and $H=$Hawk (always fights). In a contest (i.e. when a fight occurs) between two Hawks one individual will lose and incur a cost of $C$ (possibly injury). Thus payoff $E(H, H) = (V - C)/2$ as the chance of winning, and collecting the reward $V$ is $1/2$, and the chance of losing and incurring the cost $C$ is $1/2$. 
The payoff table is

<table>
<thead>
<tr>
<th></th>
<th>DOVE</th>
<th>HAWK</th>
</tr>
</thead>
<tbody>
<tr>
<td>DOVE</td>
<td>V/2</td>
<td>0</td>
</tr>
<tr>
<td>HAWK</td>
<td>V</td>
<td>(V-C)/2</td>
</tr>
</tbody>
</table>
Suppose $C < V$. Then $E(H, H) = (V - C)/2 > 0 = E(D, H)$ so D will not invade a Hawk population. Also $E(D, D) = V/2 < V = E(H, D)$ so H will invade an all D population. In fact everyone should play H.
Hawk-Dove Conflict (continued)

Now suppose $C > V$ and consider $p = (p_H, p_D) = (V/C, (C - V)/C)$. Note that $E(H, p) = (V - C)/2 * V/C + V * (C - V)/C = V(C - V)/2$ and $E(D, p) = 0 * (C - V)/C + V/2 * (C - V)/C = V(C - V)/2$.

Thus $E(H, p) = E(D, p) = E(q, p) \forall q$. It turns out that $E(q, q) < E(p, q) \forall q$ so that $p$ is an ESS. (Proofs later in a general context).
Finite/Matrix Conflicts

We introduce the payoff matrix \( A = (a_{ij}) \).

Note that \( E(x, y) = x^T A y \).

Thus for Hawk-Dove

\[
A = \begin{pmatrix}
\frac{V}{2} & 0 \\
V & \frac{(V - C)}{2}
\end{pmatrix}
\]
Some generalities (Bishop and Cannings)

We have $M$ the strategy set, define for a strategy vector/density $p$

1. $R(p) = \{x | x \in M, p_x > 0\}$, the support of $p$,
2. $S(p) = \{x | x \in M, E(x, p) = E(p, p)\}$
Theorem 2 (Bishop and Cannnings, 1976)
If $p$ is an ESS then $E(x, p) = E(p, p) \forall x \in R(p)$ except possibly for a set of measure 0.

Proof. $p$ is an ESS so $E(p, p) \geq E(x, p) \forall x \in M$. If distribution function is $P$ then we have

$$E(p, p) = \int_R E(x, p) dP(x) \leq \int_R E(p, p) dP(x) = E(p, p).$$

Comment Thus an ESS is an equilibrium over its support.

We shall ignore the pathologies and assume that

$$E(x, p) = E(p, p) \forall x \in R(p).$$
Equilibrium

Suppose that $p$ is an ESS with support $R(p)$ and $A_R$ is the sub-matrix of $A$ with indices in $R(p)$, and $p_R$ is the ESS vector restricted to the elements of $R(p)$, then we have $A_R p_R = c 1_R$ where $c$ is a constant. Then $p_R = \frac{A_R^{-1} 1_R}{1_R^T A_R^{-1} 1_R}$ and require that $p_R \geq 0$. 
Theorem 3 (Bishop and Cannings, 1976)
If $p$ and $q$ are ESS’s (wrt to same conflict) then

$$R(q) \nsubseteq S(p) \text{ and } R(p) \nsubseteq S(q)$$

Proof By contradiction. Suppose that $R(q) \subset S(p)$ then $E((p,p) = E(q,p)$. Since $p$ is an ESS this implies that $E(p,q) > E(q,q)$ which contradicts the assumption that $q$ is an ESS.
Corollary (Theorem 4 of Bishop and Cannings, 1976)
If $p$ is an ESS and $S(p) = M$ then $p$ is the unique ESS.
**Theorem** Haigh

$p$ is an ESS iff

1. $E(i, p) = E(p, p) \forall i \in R(p)$
2. $E(p, p) > E(j, p) \forall j \notin R(p)$

and

3. $z_R^T A_R z_R \leq 0$ for all $z_R$ where $z_R^T \mathbf{1}_R = 0$. Essentially $(p - q)^T A_R (p - q) \leq 0$.

NB see Abukucs (when there are some $j$ which violate (2)).
Haigh also gave the equivalent condition that if $A^* = (a_{ij} - a_{ik} - a_{ki} + a_{kk})$ for $i, j \in R(p)\{k\}$ where $k \in R(p)$, then $z^T A^* z \leq 0$, with equality iff $z = 0$, where $R(z) = R(p)\{k\}$. Thus we require $A^*$ is negative definite.
Proof
Clearly we have $E(p, p) > E(q, p)$ if $R(q) \setminus R(p) \neq \emptyset$. Thus we only need to look at $q$ with $R(q) \subset R(p)$. We require $E(p, q) > E(q, q)$ for these $q$, which is equivalent to $(E(p, p) - E(q, p) - E(p, q) + E(q, q)) = (p - q)^T A(p - q) < 0$. This corresponds to the condition (3).
The condition \((p-q)^T A (p-q) = \Delta^T A \Delta\) where \(\Delta = (\delta_1, \delta_2, \ldots, \delta_n)\) where \(\Sigma \delta_i = 0\).

We have \(\Delta^T A \Delta = \sum_{i,j=1}^{n} a_{ij} \delta_i \delta_j \)

\[= \sum_{i,j=1}^{n-1} a_{ij} \delta_i \delta_j - \sum_{i=1}^{n-1} a_{in} \delta_i (\sum_{j=1}^{n-1} \delta_j) - \sum_{i=1}^{n-1} a_{ni} \delta_i (\sum_{j=1}^{n-1} \delta_j) + a_{nn} [(\sum_{j=1}^{n-1} \delta_j)]^2 \]

\[= \sum_{i,j=1}^{n-1} (a_{ij} - a_{in} - a_{ni} + a_{nn}) \delta_i \delta_j = \Delta^T A^* \Delta \]
A $k \times k$ matrix $A$ is negative definite iff (writing the determinant of the submatrix consisting of rows 1 to $i$ as $det(i)$), $(-1)^i det(i) > 0$ for all $i$. 
Example. $A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 1 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} -14 & 8 & 10 \\ 10 & -8 & 2 \\ 2 & 8 & -6 \end{pmatrix} / 32$

so $p = \frac{A^{-1}1}{1^T A^{-1}1} = (1, 1, 1)/3$ is an equilibrium vector, and

$E(i, p) = 8/3 \forall i$, and

$A^* = \begin{pmatrix} -4 & -2 \\ -4 & -5 \end{pmatrix}$ which is negative definite.

Thus $p$ is an ESS, and since $S(p) = R$ it is the only possible ESS.
Example. \( A = \begin{pmatrix} 1 & 4 & 0 \\ 3 & 2 & 3 \\ 0 & 4 & 1 \end{pmatrix} \)

There is no appropriate equilibrium over the whole of \( M \). In fact we have \( p = (1,1,0)/2 \) gives \( 5/2 = E(1,p) = E(2,p) > E(3,p) = 2 \) and this is an ESS (check the \( A_{[1,2]} \) matrix). Also \( q = (0,1,1)/2 \) is an ESS.

In fact these are all the ESS’s. It is of interest to observe that there may be more than one ESS for a given conflict.
The set of possible ESS's for $n = 3$.

We can use the theorems to imply that the possible co-existing ESS's can have supports

- $\{1, 2, 3\}$
- $\{1, 2\}, \{1, 3\}, \{2, 3\}$
- $\{1\}, \{2, 3\}$
- $\{1\}, \{2\}\{3\}$

or any subset of these, including NO ESS. It is in fact the case that all of these are possible except ($\{1, 2\}, \{1, 3\}, \{2, 3\}$).
Given payoff matrix $A$ then the ESS’s are precisely the same as those of a matrix $A^\dagger$ where $a_{ij}^\dagger = a_{ij} + d_j$ for any set of $d_j$. Thus we could always make the $a_{ij}^\dagger = 0$. 
Suppose \( A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \)

Now we reduce this to \( A^\dagger = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix} \)
Consider the possibility of an ESS $p$ with support $R(p) = \{1, 2\}$. We have $A_R = \begin{pmatrix} 0 & a \\ c & 0 \end{pmatrix}$, $A_R^{-1} = \begin{pmatrix} 0 & a \\ c & 0 \end{pmatrix} / ac$ so a candidate ESS is $(a, c, 0)/(a + c)$ which requires $a$ and $c$ of the same sign and $A_R^\dagger = (-(a + c)/ac)$ so for an ESS we need $a > 0$. 
The strategy \((a, c, 0)/(a + c)\) is an ESS iff \(a > 0\), \(c > 0\) and 
\[ E(1, p) = E(2, p) = ac/(a + c) \geq E(3, p) = (ea + fc)/(a + c) \text{ i.e. iff} \]
\[ ac \geq ea + fc \iff 1 \geq e/c + f/a. \] For there to be three ESS’s with supports of size two we would require all the off-diagonal entries positive, and \(1 \geq e/c + f/a\), \(1 \geq b/d + a/f\) and \(1 \geq d/b + c/e\). From the first \(1 \geq e/c + f/a \Rightarrow 1 > e/c \Rightarrow c/e > 1 \Rightarrow c/e + d/b > 1\), violating the third inequality.

More generally (i.e. when \(|M| > 3\) we cannot have three pairwise ESS’s from any triple; in fact we cannot have three ESS’s with supports \(\{H \cup \{1, 2\}, \{H \cup \{1, 3\}, \{H \cup \{2, 3\}\}, \text{ where } H \subset M 1, 2, 3\}.\)
Matrix Conflicts. Possible Patterns.

Cannings, Vickers and Broom (in various combinations) have investigated the possible supports of ESS’s which can co-occur, termed patterns. There is a conjecture that if a pattern is attainable for some $|M| = n$ then so is any sub-pattern (Vickers and Cannings). This is still open, though Broom has proved that the sub-pattern is attainable for some $|M^*| > n$. 
Matrix Conflicts. Possible Patterns.
The War of Attrition, Maynard-Smith and Price (1974)

Suppose $R = [0, \infty)$, and $E(x, y) = \begin{cases} V - y & \text{if } x > y \\ V/2 - x & \text{if } x = y \\ -x & \text{if } x < y \end{cases}$

The idea is that the individuals pick a time to display and leave at that time. The individual remaining collects the reward and leaves immediately. The case $x = y$ might be thought of as being resolved through errors rather than through a precise splitting of the reward.
Now we start by seeking a solution to $E(x, p) = c$ where $c$ is a constant for $R(p) = [0, \infty)$. Such a solution can have no atoms since such a solution would be beaten by one with the atom moved an infinitesimal amount to the right. Further $c = 0$ since that is the payoff to $x = 0$. Consider a solution with a differentiable distribution function $P(y)$. 
\[ E(x, p) = \int_{y<x} (V - y) dP(y) - \int_{y>x} x dP(y) = 0 \]
\[ E(x, p) = \int_{y<x} (V - y) dP(y) - \int_{y>x} x dP(y) = 0 \]

\[ \Downarrow \]

\[ \frac{dE(x, p)}{dx} = (V - x)p(x) + xp(x) - (1 - P(x)) = 0 \]
\[ E(x, p) = \int_{y<x} (V - y) \, dP(y) - \int_{y>x} x \, dP(y) = 0 \]

\[ \Downarrow \]

\[ dE(x, p)/dx = (V - x)p(x) + xp(x) - (1 - P(x)) = 0 \]

\[ \Downarrow \]

\[ p(x)/(1 - P(x)) = 1/V \]
\[ E(x, p) = \int_{y<x} (V - y) dP(y) - \int_{y>x} x dP(y) = 0. \]

\[ \downarrow \]

\[ \frac{dE(x, p)}{dx} = (V - x)p(x) + xp(x) - (1 - P(x)) = 0 \]

\[ \downarrow \]

\[ p(x)/(1 - P(x)) = 1/V \]

\[ \downarrow \]

\[ p(x) = \left[ \exp - \left( x/V \right) \right]/V \]
Thus the equilibrium on $[0, \infty)$ is just the negative exponential with rate $(1/V)$. The contest will therefore last a negative exponential time with rate $2/V$, i.e. on average $V/2$ and $E(x, p) = 0$ all $x$.

It may seem counterintuitive to fight for no reward, but the individuals are essentially converting their time into an equivalent reward of food, sex, etc.
\[ p(x) = \frac{\exp - (x/V)}{V} \] is an ESS (Bishop and Cannings, 1978). We prove \( T(r, s) = E(r, r) - E(s, r) - E(r, s) + E(s, s) \leq 0 \) with equality iff \( r = s \). Thus since \( E(p, p) = E(q, p) \) this implies \( E(p, q) > E(q, q) \) for \( q \neq p \), so \( p \) is the unique ESS. This is intuitively reasonable since the negative exponential distribution has no memory.
Theorem.

\[ T(r, s) = E(r, r) - E(s, r) - E(r, s) + E(s, s) \leq 0 \] with equality iff \( r = s \).

Proof. Now \( E(r, r) = V/2 - E(\min(R_1, R_2)) \), \( E(s, s) = V/2 - E(\min(S_1, S_2)) \), and
\[ E(s, r) + E(r, s) = V - E(\min(S, R)). \]

Thus \( T(r, s) = -E(\min(R_1, R_2)) - E(\min(S_1, S_2)) + 2E(\min(S, R)) \).
For a non-negative random variable $X$ (Feller, 1966)

$E(X) = \int (1 - F_X(u))du$ so $E(min(X, Y)) = \int (1 - F_X(u))(1 - F_Y(u))du$.

This leads to $T(p, q) = -[\int \{(1 - R(u))^2 + (1 - S(u))^2 - 2(1 - R(u))(1 - S(u))\}du]

= -\int [R(u) - S(u)]^2du \leq 0$ with equality iff $R(u) = S(u)$ except for set of measure zero.
One can generalise the W of A to the Unlabeled Ordinal Conflict (B&C, 1978) by introducing functions $f$ and $g$ such that the payoff becomes $f(y) - g(y)$ to the winner, and $-g(y)$ to the loser when $y$ is the lesser value player. [See conditions (6) in op.cit].
Problems with the concept of an

Evolutionarily
Stable
Strategy
It is not a strategy

Evolutionarily Stable Strategy

It specifies a population average.
We need to specify what individuals play. There is a set of basic strategies $M$, the pure strategies, and an individual playing a pure strategy uses the same play each, and every, time. There are mixed strategies, some vector of probabilities over the set $M$, such that each play is made by choosing a play with the appropriate probability, independently of previous choices.
It is not stable

Evolutionarily

Stable

Strategy

a dynamic is required.
We need to specify how the frequencies of the strategies change from one generation to the next, i.e. embed the process in time.
ESS’s do not depend on the set of strategies (provided they are reachable)

Suppose we have ”pure” strategy set $M$, where $|M| = n$, payoff matrix $R$ and an internal ESS $p$. Now suppose that we have a set of available strategies $M^*$, where $|M^*| = n$, each specified by a vector of probabilities of playing the pure strategies. Denote these by $u_i$ for $i \in M^*$. Suppose that the $u$ are linearly independent and that $p$ lies in the convex hull of the $u_i$, and that $U$ is a matrix whose columns are $u_i$. 
The payoff matrix for the strategies \( u_i \) is \( U^T A U \).

The vector \( v_j = Au_j \) contains the payoffs to the pure strategies against the mixed strategy \( u_j \). Thus \( u_i^T A u_j \) is the payoff to the mixed strategy \( u_i \) against \( u_j \), and is the \((i,j)\) element of the matrix \( U^T A U \), as required.
For a vector \( v \) where \( Uv = p \) we have \( U^T A U v = U^T A p = c U^T 1 = c^* 1 \), and since \( U^T A U \) is the payoff matrix for the available strategies, we have that \( v \) is an equilibrium. Moreover for some other strategy \( w \), \( E(w,v) = w^T U^T A U v = q^T A p \leq p^T A p = v^T U^T A U v \) and so on. Thus \( v \) is an ESS, and in the original specification \( p \) is equivalently an ESS.
If we had some larger set of possible strategies which were not linearly independent then $p$ would not specify a point but some higher dimensional object.
We shall see below that the dynamics of the systems may depend on the specific set of available strategies whereas, as we have seen, the ESS’s do not (provided they exist in the spaces we consider).
The Replicator Dynamic (Continuous)

Suppose that the change in the frequency of a strategy (properly specified) \( i \) say, at time \( t \) (continuous), in a population with frequencies \( x^t_i \) at time \( t \) is given by

\[
\overset{\sim}{x}^t_i = x^t_i [E(i, x) - E(x, x)]
\]

\( \overset{\sim}{\cdot} \) denotes differentiation wrt \( t \).

NB Adding a constant \( k \) to every fitness will have no effect.
The Replicator Dynamic (Continuous)

When we have a finite conflict with payoff matrix $A$ we obtain

$$x'_i = x_i^t[E(i, x) - E(x, x)] = x_i^t[(Ax)_i - x^tA^t]$$
The Replicator Dynamic (Discrete)

Suppose that the frequency of a strategy (properly specified) \( i \) say, at time \( t \) (discrete), in a population with frequencies \( x^{t-1} \) at time \( t - 1 \) is given by

\[
x_i^t = x_i^{t-1} \frac{k + E(i, x^{t-1})}{k + E(x^{t-1}, x^{t-1})}
\]

\( k \) is a constant representing the background fitness of the population. NB. Adding \( k \) does not alter the ESS’s, but may alter the dynamic.
C should be k on this slide

Discrete Dynamic
Effect of $k$ on local stability of equilibrium $p$.

Suppose $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_i, \ldots, \hat{p}_n)$ is the equilibrium strategy vector. We consider the local behaviour near $\hat{p}$. The standard method is to take a point sufficiently near the equilibrium that we can approximate by a linear system.
Given the matrix $A$ and $\hat{p}$ an equilibrium for the discrete dynamic.

Now consider a starting position $\hat{p} + \Delta$, and denote the successor value by $\hat{p} + \Delta'$. We have to expand the equation, dropping terms in $\delta_i \delta_j$ etc.

$$\hat{p}_i + \delta'_i = \frac{(\hat{p}_i + \delta_i)(\sum_j a_{ij}(\hat{p}_j + \delta_j))}{\sum_{ij} a_{ij}(\hat{p}_i + \delta_i)(\hat{p}_j + \delta_j)}$$
The result of the expansion of the above is

$$\delta'_i = \sum_j b_{ij} \delta_j$$

so

$$\Delta' = \frac{B \Delta}{\hat{W}}$$

for some matrix $B$ which is a function of the $a_{ij}$'s, and $\hat{W} = \Sigma_{ij} a_{ij} \hat{p}_i \hat{p}_j$. Note that $\Sigma_i \delta_i = 0$. We need to examine the eigenvalues of the reduced form of $B/\hat{W} = D$ (i.e. take $d_{ij} = (b_{ij} - b_{in} - b_{nj} + b_{nn})/\hat{W}$).
Now suppose we consider a matrix of payoffs where each $a_{ij}$ is replaced by $k + a_{ij}$. Now the linearised system gives

$$\Delta' = \frac{(kI + B)\Delta}{(k + \hat{W})}.$$  Now when we reduce this form we obtain $D^*$ say, where $D^* = D / (k + \hat{W})$. If $k + \hat{W} > 0$ then as we increase $k$ we reduce the sizes of the eigenvalues (the eigenvectors do not change, and at some stage they will all have modulus less than unity, so convergence will be assured.
Rock, Scissors, Paper
Rock-Scissors-Paper, Gary Larson.
Three colors Red Blue Green. Red beats Blue; Blue beats Green; Green beats Red.
Rock-Scissors-Paper

Payoffs

\[
\begin{bmatrix}
0 & +1 & -1 \\
-1 & 0 & +1 \\
+1 & -1 & 0
\end{bmatrix}
\]

NB There is an equilibrium \( p = (1, 1, 1)/3 \), but this is not an ESS since \( E(q, p) = E(p, p) = E(p, q) = E(q, q) = 0 \), also \((1, 0, 0)\) etc.
Cyclic Payoffs

\[ A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \]

Equilibrium at \((1, 1, 1)/3\).

ESS. We have to examine the negative definiteness of

\[ A = \begin{bmatrix} 2a - b - c & a + b - 2c \\ a + c - 2b & 2a - b - c \end{bmatrix} \]

and this is negative definite when \(2a < (b + c)\).
Cyclic Payoffs; Replicator Dynamic, continuous time.

Suppose we have payoff matrix \( A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \)

Suppose the frequencies of the three strategies are \( x^t = (x_1^t, x_2^t, x_3^t) \). We prove that \( x_1^t x_2^t x_3^t \) is a monotone function through time.
NB

\[ \sum_i [(Ax)^t_i] = (ax_1^t + bx_2^t + cx_3^t) + (cx_1^t + ax_2^t + bx_3^t) + (bx_1^t + cx_2^t + ax_3^t) = (a + b + c)(x_1^t + x_2^t + x_3^t) = (a + b + c) \]

and

\[ x^t Ax^t = a(x_1^{t2} + x_2^{t2} + x_3^{t2}) + (b + c)(x_1^tx_2^t + x_1^tx_3^t + x_2^tx_3^t) \]
Write $\dot{}$ denotes $d/dt$, and $\Sigma^2 = (x_1^t + x_2^t + x_3^t)$.

Now consider $f^t = \prod_i x_i^t$ then

$$\dot{f^t} = \Sigma_i x_i^t f^t / x_i^t = f^t \Sigma_i [(Ax)_i^t - x'^t Ax^t] = f^t [(a + b + c) - 3a\Sigma^2 - 3(b + c)(1 - \Sigma^2)/2] = f^t (a - (b + c)/2)(1 - 3\Sigma^2).$$
\[ f^t(a - (b + c)/2)(1 - 3\Sigma^2). \]

Now \( \Sigma^2 \in [1/3, 1] \) so \( (1 - 3\Sigma^2) \in [-2, 0] \). Thus \( f^t \) is increasing, constant or decreasing (with \( t \)) according as \( 2a < (b + c) \) is negative, zero or positive. System therefore spirals in, cycles or spirals out as \( 2a < (b + c) \) is negative, zero or positive, i.e. it converges to the internal ESS if that exists.
Mixed Strategies in the Cyclic Model, continuous replicator dynamic.

Suppose that we have a set of cyclic mixed strategies, \( y_1 = (u, v, w) \), \( y_2 = (w, u, v) \), and \( y_3 = (v, w, u) \) (i.e. the first plays rock, scissors, paper with probabilities \( u \), \( v \) and \( w \)). Then the payoff matrix is \( A^* = YAY^T \) where \( Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \).
Of course $A^*$ is cyclic, in fact

$$A^* = \begin{bmatrix}
  a\delta + (b + c)\tau & b\delta + (a + c)\tau & c\delta + (a + b)\tau \\
  c\delta + (a + b)\tau & a\delta + (b + c)\tau & b\delta + (a + c)\tau \\
  b\delta + (a + c)\tau & c\delta + (a + b)\tau & a\delta + (b + c)\tau
\end{bmatrix}$$

where $\delta = u^2 + v^2 + w^2$ and $\tau = yv + vw + wu$. So the previous analysis can be applied and the critical expression is $(2a - (b + c))(\delta - \tau)$, and since $\delta \geq \tau$ with equality iff $u = v = w = 1/3$ the asymptotic behaviour is the same as for the pure strategy case (i.e. goes to centre, edge or cycles). ESS condition is the same. Speed will be different.
RSP, Replicator Dynamic, discrete time.
Now \( k \) is of importance. In fact \( \Delta_t = \frac{x'_t A x^t}{x^t_1 x^t_2 x^t_3} \) plays the same role as \( f^t \). In terms of our \( a, b \) and \( c \) the conditions become \( a^2 - bc < 0 \) spirals in, etc.
RSP. Discrete replicator dynamic, mixed strategies.

As before take symmetric strategies \((u, v, w)\) etc. Then we had

\[
A^* = \begin{bmatrix}
    a\delta + (b + c)\tau & b\delta + (a + c)\tau & c\delta + (a + b)\tau \\
    c\delta + (a + b)\tau & a\delta + (b + c)\tau & b\delta + (a + c)\tau \\
    b\delta + (a + c)\tau & c\delta + (a + b)\tau & a\delta + (b + c)\tau 
\end{bmatrix}
\]

Analysis for ESS is of course unchanged by consideration of a dynamic. There is an ESS with cyclic mixture iff ESS with pures.
Then we have convergence to the centre iff $(a^*)^2 - b^*c^* < 0$ i.e. $0 > (a\delta + (b + c)\tau)^2 - (b\delta + (a + c)\tau)(c\delta + (a + b)\tau) \equiv (a^2 - bc)\delta^2 - ((b^2 - ac) + (c^2 - ab))\delta\tau + ((b^2 - ac) + (c^2 - ab) - (a^2 - bc))\tau^2 = (\alpha\delta^2 - (\beta + \gamma)\delta\tau + (\beta + \gamma - \alpha)\tau^2) = (\alpha\delta - (\beta + \gamma - \alpha)\tau)(\delta - \tau)$ where $\alpha = (a^2 - bc)$, $\beta = (b^2 - ac)$ and $\gamma = (c^2 - ab)$. 
Now $\delta \geq \tau$ with equality iff $u = v = w = 1/3$, so convergence requires $0 > (\alpha \delta - (\beta + \gamma - \alpha)\tau)$. Since $\delta + 2\tau = 1$ we obtain $0 > \delta(\alpha + \beta + \gamma) - (\beta + \gamma - \alpha) \Rightarrow (\beta + \gamma - \alpha) > \delta(\alpha + \beta + \gamma)$. 
The RSP model is often presented in the following form; background fitness $k$ and $A$ is (we can obtain this by a suitable choice of $a, b, c$.

\[
A = kW + \begin{bmatrix}
-\epsilon & +1 & -1 \\
-1 & -\epsilon & +1 \\
+1 & -1 & -\epsilon
\end{bmatrix}
\]

where $W$ is matrix of 1’s.

Then $A^*$ is

\[
\begin{bmatrix}
-2\epsilon & 0 \\
0 & -2\epsilon
\end{bmatrix}
\]

which is negative definite provided $\epsilon > 0$, so then there is an ESS at $(1, 1, 1)/3$. 
The earlier condition for convergence to the ESS is
$0 > (\beta + \gamma - \alpha) > \delta(\alpha + \beta + \gamma)$ and converting to the new notation
we get convergence when
$(1 + 4k\epsilon - \epsilon^2) > \delta(3 + \epsilon^2)$.

We take some examples where $\epsilon = 0.5$ so convergence to $(1, 1, 1)/3$
for puries i.e. $\delta = 1$ we require $k > (1 + \epsilon^2)/2\epsilon$ so $k > 1.25$. 
RSP, $c=1.3, \varepsilon=0.5$
RSP, \( k = 1.1, \epsilon = 0.5 \)

R, S and P. Spirals out to boundary.
For $k = 1.1$ and $\epsilon = 0.5$ we have convergence iff $(1 + 4k\epsilon - \epsilon^2) = 2.95 > \delta(3 + \epsilon^2) = 3.25\delta$ so require $\delta < 2.95/3.25$. 
\[(RS^*), (R^*P), \& (*)SP); c=1.1, \varepsilon=0.5\]
There is no theory (as far as I am aware) for other combinations of mixtures. Here are some simple examples which have been iterated. Starting with a completes set of pures plus a \((1,1,1)/3\). All examples are for \(k = 1.1, \epsilon = 0.5\). Necessarily the system will converge to \(\rho/3\) for each of R,S initial and P and to \((1 - \rho)\) for \((1,1,1)/3\), depending on the initial frequencies.
Pures & (R,S,P)/3, k=1.1, ε=0.5
Pures & \( (R,S,P)/3 \), \( k=1.1, \varepsilon=0.5 \)

Freq of Rock & \( (R,S,P)/3 \)

Time
Pures & \((R,S,P)/3\), \(k=1.1, \varepsilon=0.5\)

Freq of Rock & \((R,S,P)/3\)
Note that the frequency of the mixture always increases monotonically.
A single example of a non-symmetric case. \((0,1,1)/2\) plus the pures. The only equilibrium is \((1,1,1)/3\) and the only way to obtain that is \((2/3)((1,1,0)/2) + (1/3)(0,0,1)\)
\{S,P\}, R, S \ k=1.1, \epsilon=0.5
An Interior ESS is "globally stable".

If we have an ESS $p$ over the whole $S$ then it is unique. If the population frequency is $q$ then the fitness of the ESS strategy is $E(p, q)$, and since it is an ESS we know that $E(q, p) = E(p, p)$ and $E(p, q) > E(q, q)$, so the ESS strategy is fitter than the average fitness of the population ($E(q, q)$). Thus for any sensible dynamic (i.e. one for which strategies whose fitness is above the average at time $t$ have increased frequency at time $t + 1$) an individual playing the ESS strategy will increase in frequency until the equilibrium is reached.
Finally wrt RSP we have an example where the frequencies are subject to perturbations (i.e. we have a stochastic system). Note that the \((1, 1, 1)/3\) strategy goes to a value near 1.
Pures + (R,S,P)/3, k=1.1, ε=0.5, Random perturbations

Freq of Rock & (R,S,P)/3
Pures + \((R,S,P)/3\), \(k=1.1, \epsilon=0.5\),
Random Perturbations
R,S,P and (R,S,P)/3
The Dynamics of the War of Attrition.

Suppose that only pure strategies can be played. Start with just two strategies \( m_{n-1} < m_n \). We look at a general payoff matrix

\[
A = \begin{pmatrix} a & g \\ h & b \end{pmatrix}
\]
Consider the discrete dynamic so take strategy frequencies $p^t$ and $q^t$ then $p^{t+1} = p^t(ap^t + gq^t)/W^t$ and $q^{t+1} = p^t(hp^t + bq^t)/W^t$ where $W^t = a(p^t)^2 + (g + h)p^t q^t + b(q^t)^2$ the average payoff. Then if $f^t = p^t/q^t$ we have

$$f^{t+1} = f^t(af^t + g)/(hf^t + b)$$
Now note that the equilibria, given where $f^t = f^{t+1}$ are $0, \infty$ and $(h - b)/(g - a)$ and $df^{t+1}/df^t > 0$. There is thus an interior fixed point iff $(h - b)$ and $(g - a)$ have the same parity.
For \((h - b) > 0\) and \((g - a) > 0\) we have
\[
f^t > (h - b)/(g - a) \Rightarrow f^t(g - a) > (h - b) \Rightarrow gf^t + b > af^t + h \Rightarrow f^{t+1} < f^t.
\]
Similarly
\[
f^t < (h - b)/(g - a) \Rightarrow f^{t+1} > f^t.
\]
Using this plus the positive derivative we have that convergence is monotone, and to an interior point \(f = (h - b)/(g - a)\) iff \(h > b\) and \(g > a\).
For the W of A we have

\[ A = \begin{pmatrix} \frac{V}{2} - m_{n-1} & -m_{n-1} \\ V - m_{n-1} & V - m_n \end{pmatrix} \]

and so in the earlier notation \( g > a \). There is an interior equilibrium at \( \frac{(h - b)}{(g - a)} = \frac{[m_n - m_{n-1} - V]}{V/2} \) iff \( m_n - m_{n-1} > V \) i.e. there is a big enough gap. Otherwise the system converges monotonically to all \( m_n \).
For $n$ strategies

$$A = \begin{pmatrix}
V/2 - m_1 & -m_1 & m_1 & \ldots & -m_1 & m_1 & m_1 \\
V - m_1 & V/2 - m_2 & -m_2 & \ldots & m_2 & m_2 & m_2 \\
V - m_1 & V - m_2 & V/2 - m_3 & \ldots & -m_3 & -m_3 & m_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
V - m_1 & V - m_2 & V/2 - m_3 & \ldots & V/2 - m_{n-2} & -m_{n-2} & m_{n-2} \\
V - m_1 & V - m_2 & V/2 - m_3 & \ldots & V - m_{n-1} & V/2 - m_{n-1} & -m_{n-1} \\
V - m_1 & V - m_2 & V/2 - m_3 & \ldots & V - m_{n-2} & V - m_{n-1} & V/2 - m_n
\end{pmatrix}$$
For $n$ strategies

$$A = \begin{pmatrix}
V/2 - m_1 & -m_1 & m_1 & \ldots & -m_1 & m_1 & m_1 \\
V - m_1 & V/2 - m_2 & -m_2 & \ldots & m_2 & m_2 & m_2 \\
V - m_1 & V - m_2 & V/2 - m_3 & \ldots & -m_3 & -m_3 & m_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
V - m_1 & V - m_2 & V/2 - m_3 & \ldots & V/2 - m_{n-2} & -m_{n-2} & m_{n-2} \\
V - m_1 & V - m_2 & V/2 - m_3 & \ldots & V - m_{n-1} & V/2 - m_{n-1} & -m_{n-1} \\
V - m_1 & V - m_2 & V/2 - m_3 & \ldots & V - m_{n-2} & V - m_{n-1} & V/2 - m_n
\end{pmatrix}$$
Noting that the entries in the columns 1 to $i - 1$ are identical for rows $i$ to $n$ we see that if there is an equilibrium over the set $T = \{k, k + 1, \ldots, n - 1, n\}$ with frequencies $p_i$ say, then any equilibrium over a set $W \supset T$ must have frequencies $\lambda p_i$ for all $i \in T$. 
We can find the Equilibria, the ESS’s and prove that the system always converges to the unique ESS by working sequentially from $m_n$ to $(m_n, m_{n-1})$ to $(m_n, m_{n-1}, m_{n-2})$ and so on.
Suppose the system has converged to the equilibrium \( p \) over \( \{m_{k+1}, m_{k+2}, \ldots, m_n\} \) and now consider the behaviour of \( m_k \) vis-a-vis wrt this. We have

\[
E(m_k, p) = -m_k
\]

and

\[
E(m_{k+1}, p) = E(p, p) = Vp_{k+1}/2 - m_{k+1}
\]

so \( m_k \) invades if \( W = (m_{k+1} - m_k) - Vp_{k+1} > 0 \) and its frequency converges monotonically to \( p_k = W/(W + V/2) \) as the other frequencies converge monotonically to \( p/(1 - p_k) \).
Now the system converges to the equilibrium over the enlarged set \( \{m_k, m_{k+1}, m_{k+2}, \ldots, m_n\} \)

possibly without \( m_k \).
The system now converges monotonically as far as the \( \{m_k, m_{k+1}, m_{k+2}, \ldots, m_n\} \) are concerned. We then consider the next value down \( m_{k-1} \). Provided the system has got sufficiently close to its target we can use the same argument and so on. Convergence is assured.

Note that adding any \( k \) makes no difference.
W of A with finite time limit

Often the length of contest will be limited (length of day perhaps). Then the solution is given by
Finite Interval \([0, m]\)

- Neg. Exp. over \([0, m-v/2]\) and Atom at \(m\).
- \(E(p,p)=0\)
Discrete Space

- Suppose $S=\{m_0, m_1, m_2, \ldots, m_{k-1}, m_k\}$
  where $m_i < m_{i+1}$ all $i$. Then (we revisit later)
  obtain a unique ESS, with atoms on a subset of $S$, e.g.
Suppose we have three pure strategies $m_2 > m_1 > m_0$ which are such that there is an ESS, $p$ say, with all present. Then $E(m_i, p) = c$. The fitness of some other $x$-value introduced into a population at the ESS $p$ will be given by

$$E(x, p) = \begin{cases} 
-x & \text{if } x < m_0 \\
p_0(V/2 - m_0) - m_0(1 - p_0) & \text{if } x = m_0 \\
p_0(V - m_0) - x(1 - p_0) & \text{if } m_0 < x < m_1 \\
p_0(V - m_0) + p_1(V/2 - m_1) - xp_2 & \text{if } x = m_1 \\
p_0(V - m_0) + p_1(V - m_1) - xp_2 & \text{if } m_1 < x < m_2 \\
p_0(V - m_0) + p_1(V - m_1) + p_2(V/2 - m_2) & \text{if } x = m_2 \\
p_0(V - m_0) + p_1(V - m_1) + p_2(V - m_2) & \text{if } m_2 < x 
\end{cases}$$

so note that this is piecewise linear with jumps $Vp_i$ to left and right of each $m_i$. The gradients of the linear pieces are $\leq 0$ and steeper to the left, $(-1, -(1-p_0), -(1-p_0-p_1) = p_2, 0)$ respectively.
Payoffs: Discrete $S=\{m_0 < m_1 < m_2\}$
War of Attrition, $V = 10$; Highest Value in Population, $m \in [0, 10]$

Time $t = 0 - 1000$
War of Attrition $V = 10$; Highest, second highest and lowest values

Time $t = 0 - 100$
War of Attrition $V = 10$; Highest, second highest and lowest values

Time $t = 0 - 1000$
War of Attrition $V = 10$; Frequency of highest value,

Time $t = 0 - 1000$
Frequency of 10 when $V = 10$ is $\exp\left(-\frac{m-V/2}{V}\right) = \exp(-0.5) = 0.606$
War of Attrition, $V = 10$, Number of Strategies.

Time $t = 0 - 1000$
War of Attrition, $V = 10$, Number of Strategies.
War of Attrition, $V = 10$, Payoff $E(p, p)$.

Time $t = 0 - 1000$
War of Attrition, $V = 10$, Frequencies $[0, 5]$.

Time $t = 1,000$

Time $t = 10,000$
War of Attrition, $V = 10$, Frequencies $[0, 5]$.

Time $t = 100,000$