Ancestry in the face of competition, v0.1:
Directed random walk on the directed percolation cluster

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Based on joint work in progress with J. Černý,
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Remark. The catchier part of the title is due to Steve Evans, who invented it in Oberwolfach in August 2005.
General aim:
Study/understand the space-time embedding of ancestral lineages in spatial models for populations with local density regulation (in particular, with non-constant local population sizes).
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1 Directed percolation

2 Random walk on the cluster
   - A renewal structure

3 Locally regulated populations (and ancestral lineages)
Directed (site) percolation

\[ p \in (0, 1), \omega(x, n), x \in \mathbb{Z}^d, n \in \mathbb{Z}\text{ i.i.d. Bernoulli}(p). \]

Interpretation: \( \omega(x, n) = 1 \): site \((x, n)\) is open, otherwise closed
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\( m < n, \ x, y \in \mathbb{Z}^d : (x, m) \to (y, n) \) if there exist \( x = x_0, x_1, \ldots, x_{n-m} = y \) such that \( x_i - x_{i-1} \in U \) and \( \omega(x_i, m + i) = 1 \) for \( i = 1, \ldots, n - m \) and
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\( C_0 := \{(y, n) : y \in \mathbb{Z}^d, n \geq 0, (0, 0) \rightarrow (y, n)\} \) is the (directed) cluster of the origin
There exists $p_c \in (0, 1)$ such that

$$\mathbb{P}(|C_0| = \infty) > 0 \text{ iff } p > p_c.$$
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If $p > p_c$, $\mathbb{P}(C_0 \text{ reaches height } n \mid |C_0| < \infty) \leq C e^{-cn}$ for some $c, C \in (0, \infty)$. 

M. Birkner (JGU Mainz)
The discrete time contact process and directed percolation

\[ \eta_n(x), \ n \in \mathbb{Z}_+, \ x \in \mathbb{Z}^d \] with values in \( \{0, 1\} \).

Site \( x \) is generation \( n \) is “inhabited” (or: “infected”) if \( \eta_n(x) = 1 \).
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Dynamics: \( U \subset \mathbb{Z}^d \) finite, symmetric, \( p \in (0, 1) \).
Given \( \eta_n \), independently for \( x \in \mathbb{Z}^d \),

\[
\eta_{n+1}(x) = \begin{cases} 
1 & \text{w. prob. } p \cdot 1(\eta_n(y) = 1 \text{ for some } y \in x + U) \\
0 & \text{w. prob. } 1 - p \cdot 1(\eta_n(y) = 1 \text{ for some } y \in x + U)
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\( \eta_n(x) = 1 \) iff \( (y, 0) \rightarrow (x, n) \) for some \( y \in \mathbb{Z}^d \) with \( \eta_0(y) = 1 \).
The discrete time contact process

Self duality: For $A, B \subset \mathbb{Z}^d$

$$\mathbb{P}(\eta_n(x) = 0 \text{ for all } x \in B \mid \eta_0(\cdot) = 1_A(\cdot))$$

$$= \mathbb{P}(\eta_n(x) = 0 \text{ for all } x \in A \mid \eta_0(\cdot) = 1_B(\cdot))$$
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\]

Stationary process:
For $p > p_c$, there is a (unique extremal) non-trivial stationary distribution. Informally, $\eta_0^{\text{stat}}(x) = 1$ iff $\mathbb{Z}^d \times \{-\infty\} \rightarrow (x, 0)$
The discrete time contact process ...

... as a locally regulated population model

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\end{cases} \]

Possible interpretation for ancestry:
In generation \( n + 1 \), each site \( x \) is inhabitable with probability \( p \)
If \( \eta_n(y) = 1 \) of some \( y \in x + U \), the particle at \( y \) in gen. \( n \) puts an offspring at \( x \).
If several \( y \) are eligible, one is chosen at random.
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If several \( y \) are eligible, one is chosen at random.
Thus, individuals in “sparsely populated” regions have a higher reproduction probability.
An ancestral line in the discrete time contact process

\( p > p_c, (\eta_n^{\text{stat}}(x), x \in \mathbb{Z}^d, n \in \mathbb{Z}) \) stationary DCP, assume \( \eta_0^{\text{stat}}(0) = 1 \).

Let \( X_n \) = position of the ancestor of the individual at the (space-time) origin \( n \) generations ago.
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Let $X_n =$ position of the ancestor of the individual at the (space-time) origin $n$ generations ago. Given $\eta^{\text{stat}}$ and $X_n = x$, $X_{n+1}$ is uniform on 

$$\{y \in \mathbb{Z}^d : y - x \in U, \eta_{-n-1}^{\text{stat}}(y) = 1\} \ (\neq \emptyset).$$
Directed percolation

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\[ \{ y \in \mathbb{Z}^d : y - x \in U, \eta_{-n-1}^{\text{stat}}(y) = 1 \} \quad (\neq \emptyset). \]

To avoid lots of \(-\)-signs later, put \( \xi_n(x) := \eta_{-n}^{\text{stat}}(x), x \in \mathbb{Z}^d, n \in \mathbb{Z} \).

Note: \( \xi_n(x) = 1 \) iff \((x, n) \to \mathbb{Z}^d \times \{+\infty\}\)
Directed random walk on the supercritical directed cluster

$$\omega(x, n), x \in \mathbb{Z}^d, n \in \mathbb{Z}, \text{ i.i.d. Bernoulli}(p), p > p_c$$

$$\xi_n(x) = 1 \text{ iff } (x, n) \to (y, k) \text{ for infinitely many } (y, k) \quad \text{ (“}(x, n) \to +\infty\text{”) }$$

Write \( C := \{(y, m) : \xi_m(y) = 1\} \), \( U(x, n) := (x + U) \times \{n + 1\} \).
Random walk on the cluster

Directed random walk on the supercritical directed cluster

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Let \[ X_0 = 0 (\in \mathbb{Z}^d), \]

\[ \mathbb{P}(X_{n+1} = y \mid \xi, X_n = x, X_{n-1} = x_{n-1}, \ldots X_1 = x_1) = \frac{1(y \in U(x, n) \cap C)}{|U(x, n) \cap C|} \]

(with some arbitrary setting if \[ U(x, n) \cap C = \emptyset, \text{ we will later consider } \xi \text{ under } \mathbb{P}(\cdot \mid (0, 0) \in C) \])
Random walk on the cluster

Directed random walk on the supercritical directed cluster

\( \omega(x, n), x \in \mathbb{Z}^d, n \in \mathbb{Z}, \) i.i.d. Bernoulli(\( p \)), \( p > p_c \)

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Aim: Understand the long-time behaviour of \((X_n)\). Is it similar to "ordinary" random walk?
Remark.

$(X_n)$ is a random walk in space-time random environment (given by $\xi$).

Random walks in random environments and recently also random walk in space-time random environments have received considerable attention (see e.g. Firas Rassoul-Agha’s homepage http://www.math.utah.edu/~firas/Research/)

As far as we know, none of the general techniques developed so far in this context is applicable:

- $(X_n)$ is not uniformly elliptic.
- $\xi$ is complicated: not i.i.d., nor is $(\xi_n(x))_{n=0,1,...}$ for fixed $x$ a Markov chain.
- The abstract conditions from Dolgopyat, Keller and Liverani (2008) appear very hard to verify.
Ancestor ordering

For \( x \in \mathbb{Z}^d \), \( n \in \mathbb{Z} \) let \( \tilde{\omega}(x, n) = (\tilde{\omega}(x, n)[1], \tilde{\omega}(x, n)[2], \ldots, \tilde{\omega}(x, n)[|U|]) \) an independent uniform permutation of \( U(x, n) = (x + U) \times \{n + 1\} \).
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$\Gamma^k_{(x, n)}$ := set of all $k$-step (directed) paths
\[ \gamma = ((x_0, n), (x_1, n + 1), \ldots, (x_k, n + k)) \]
starting at $x_0 = x$ whose steps begin at open sites, i.e., $\omega(x_i, n + i) = 1$ for $i = 0, 1, \ldots, k - 1$. 

Remarks.
1) Construction measurable w.r.t. $\sigma(\omega(y, i), \tilde{\omega}(y, i) : y \in \mathbb{Z}^d, n \leq i < n + k)$
2) Discrete time analogue of Neuhauser (1992)
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Order \( \Gamma^k_{(x, n)} \ni \gamma, \gamma' = ((x = x'_0, n), (x'_1, n + 1), \ldots, (x'_k, n + k)) \):

1 \( \leq \ell (\prec k) \) the minimal value s.th. \( x_\ell \neq x'_\ell \), then

\[ \gamma \prec \gamma' \text{ if } x_\ell \text{ has a smaller index than } x'_\ell \text{ in } \tilde{\omega}(x_{\ell-1}, n + \ell - 1). \]

\[ A^{(1)}_{(x, n); k} := \text{(spatial) endpoint of the smallest path in } \Gamma^k_{(x, n)} \text{ (if } \Gamma^k_{(x, n)} \neq \emptyset) \]

(first (potential) ancestor \( k \) generations ago of site \( (x, n) \))
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\( A^{(1)}_{(x,n);k} := \) (spatial) endpoint of the smallest path in \( \Gamma^k_{(x,n)} \) (if \( \Gamma^k_{(x,n)} \neq \emptyset \))
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Remarks. 1) Construction measurable w.r.t.
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2) Discrete time analogue of Neuhauser (1992)
Ancestor ordering and regeneration

\[ \kappa(x, n) := \tilde{\omega}(x, n) \left[ \min\{ i : \xi_{n+1}(\tilde{\omega}(x, n)[i]) = 1 \} \wedge |U| \right] \]  
(with min \( \emptyset := +\infty \))

\( \kappa(x, n) \) is uniformly distributed on \( U(x, n) \cap C \) if the latter is not empty and uniformly distributed on \( U(x, n) \) otherwise.
Ancestor ordering and regeneration

\[ \kappa(x, n) := \bar{\omega}(x, n) \left[ \min \{ i : \xi_{n+1}(\bar{\omega}(x, n)[i]) = 1 \} \wedge |U| \right] \text{ (with } \min \emptyset := +\infty) \]

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On \( A_0 := \{(0, 0) \in C\} \)

\[ X_0 = 0, \quad X_{n+1} := \kappa(X_n, n), \quad n = 1, 2, \ldots \]

is (a version of) the directed random walk on \( C \), and \( X_k = A_{(0,0);k}^{(1)} \) if \( \xi_k(A_{(0,0);k}^{(1)}) = 1 \).
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**Regeneration times:**

\( T_0 = 0, \ Y_0 = 0, \)

\( T_1 = \min\{n > 0 : \xi_n(A^{(1)}_{(0,0);n}) = 1\}, \ Y_1 = A^{(1)}_{(0,0);n}, \)

then \( T_2 = \min\{n > 0 : \xi_{T_1+n}(A^{(1)}_{(Y_1, T_1);n}) = 1\}, \) etc.
Proposition

\((Y_i - Y_{i-1}, T_i - T_{i-1}))_{i \geq 1}\) is i.i.d. under \(\mathbb{P}(\cdot \mid A_0)\), \(Y_1\) is symmetrically distributed. There exist \(C, c \in (0, \infty)\), such that

\[
\mathbb{P}(\|Y_1\| > n \mid A_0), \mathbb{P}(\tau_1 > n \mid A_0) \leq Ce^{-cn} \text{ for } n \in \mathbb{N}.
\]
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\((Y_i - Y_{i-1}, T_i - T_{i-1}))_{i \geq 1}\) is i.i.d. under \(P(\cdot | A_0)\), \(Y_1\) is symmetrically distributed. There exist \(C, c \in (0, \infty)\), such that

\[ P(||Y_1|| > n | A_0), P(\tau_1 > n | A_0) \leq Ce^{-cn} \text{ for } n \in \mathbb{N}. \]

Remark

Regeneration structure and proof analogous to Kuczek (1989) and adaptation by Neuhauser (1992):

For tails of \(T_1 - T_0\) use “restart” argument (to remove conditioning on \(A_0\)) and the fact that finite clusters are small, i.i.d. property follows from the fact that the ancestor ordering construction uses disjoint time-slices.
LLN and annealed CLT for directed walk on the cluster

Corollary

$$\mathbb{P}
\left(
\frac{1}{n} X_n \to 0 \mid \omega
\right)
= 1 \quad \text{for } \mathbb{P}(\cdot \mid A_0)-\text{a.a. } \omega, \text{ and}

\lim_{n \to \infty}
\mathbb{P}
\left(
\frac{1}{\sqrt{n}} X_n \leq x \mid A_0
\right)
= \Phi(x) \quad \text{for } x \in \mathbb{R}^d,$$

with $\Phi$ the distribution function of a non-trivial $d$-dimensional normal law.
Two walks on the same cluster

\((X_n), (X'_n)\) two independent directed walks on the same supercritical directed cluster (i.e. using the same \(\omega\)'s, but independent \(\tilde{\omega}\)'s resp. \(\tilde{\omega}'\).

**Hopeful theorem in progress ...**

\[
\lim_{n \to \infty} \mathbb{P}\left( \frac{1}{\sqrt{n}} X_n \leq x, \frac{1}{\sqrt{n}} X_n \leq x' \mid A_0 \right) = \Phi(x) \Phi(x') \quad \text{for } x, x' \in \mathbb{R}^d,
\]

which implies \(\mathbb{P}\left( \frac{1}{\sqrt{n}} X_n \leq x \mid \omega \right) \to \Phi(x) \quad \text{in } L^2(\mathbb{P}(\cdot \mid A_0)).\)
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**Remarks**

1) Quantitative strengthening may allow an a.s. CLT for \((X_n)\)
2) Variation where \((X_n)\) and \((X'_n)\) coalesce upon meeting is of (great) interest in mathematical population genetics
3) (Some) analogous arguments for the continuous-time case by Neuhauser (1992) and Valesin (2010).
A spatial logistic model

Particles “live” in $\mathbb{Z}^d$ in discrete generations,
\[ \eta_n(x) = \# \text{ particles at } x \in \mathbb{Z}^d \text{ in generation } n. \]

Given $\eta_n$,

each particle at $x$ has Poisson\((m - \sum_z \lambda_{z-x} \eta_n(z)))_+\) offspring,
\[ m > 1, \lambda_z \geq 0, \lambda_0 > 0, \text{ finite range}. \]

Children take an independent random walk step to $y$ with probability $p_{y-x}$,
\[ p_{xy} = p_{y-x} \text{ symmetric, aperiodic finite range random walk kernel on } \mathbb{Z}^d. \]
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$m > 1, \lambda_z \geq 0, \lambda_0 > 0$, finite range.
Children take an independent random walk step to $y$ with probability $p_{y-x}$, 
$p_{xy} = p_{y-x}$ symmetric, aperiodic finite range random walk kernel on $\mathbb{Z}^d$.

Given $\eta_n$,

$$\eta_{n+1}(y) \sim \text{Poi} \left( \sum_x p_{y-x} \eta_n(x) \left( m - \sum_z \lambda_{z-x} \eta_n(z) \right)_+ \right), \quad \text{independent}$$
Survival and complete convergence

Theorem (B. & Depperschmidt, 2007)

Assume $m \in (1, 3)$, $0 < \lambda_0 \ll 1$, $\lambda_z \ll \lambda_0$ for $z \neq 0$.

$(\eta_n)$ survives for all time globally and locally with positive probability for any non-trivial initial condition $\eta_0$.

Given survival, $\eta_n$ converges in distribution to its unique non-trivial equilibrium.

Starting from any two initial conditions $\eta_0, \eta'_0$, copies $(\eta_n)$, $(\eta'_n)$ can be coupled such that if both survive, $\eta_n(x) = \eta'_n(x)$ in a space-time cone.
Survival and complete convergence

**Theorem (B. & Depperschmidt, 2007)**

Assume $m \in (1, 3)$, $0 < \lambda_0 \ll 1$, $\lambda_z \ll \lambda_0$ for $z \neq 0$.

$(\eta_n)$ survives for all time globally and locally with positive probability for any non-trivial initial condition $\eta_0$.

Given survival, $\eta_n$ converges in distribution to its unique non-trivial equilibrium.

Starting from any two initial conditions $\eta_0$, $\eta_0'$, copies $(\eta_n)$, $(\eta_n')$ can be coupled such that if both survive, $\eta_n(x) = \eta_n'(x)$ in a space-time cone.

Proof uses that corresponding deterministic system

$$
\zeta_{n+1}(y) = \sum_x p_{y-x}\zeta_n(x)\left(m - \sum_z \lambda_z \zeta_n(z)\right)_+
$$

has unique non-triv. fixed point

plus coarse-graining, lots of comparisons with directed percolation.
Coupling

$\mathbf{m} = 1.5, \quad \mathbf{p} = (1/3, 1/3, 1/3), \quad \lambda = (0.01, 0.02, 0.01)$
$m = 1.5, \ p = (1/3, 1/3, 1/3), \ \lambda = (0.01, 0.02, 0.01)$
Locally regulated populations (and ancestral lineages)

Coupling

\[ m = 1.5, \ p = (1/3, 1/3, 1/3), \ \lambda = (0.01, 0.02, 0.01) \]
Ancestral lines

Given stationary ($\eta_{n}^{\text{stat}}(x), n \in \mathbb{Z}, x \in \mathbb{Z}^d$), cond. on $\eta_0^{\text{stat}}(0) > 0$, sample an individual from space-time origin (0, 0) (uniformly)

Let ($X_n$) position of her ancestor $n$ generations ago:

Given $\eta^{\text{stat}}$ and $X_n = x$, $X_{n+1} = y$ w. prob.

$$p_{x-y}\eta_{-n-1}^{\text{stat}}(y) \left( m - \sum_z \lambda_{z-y}\eta_{-n-1}^{\text{stat}}(z) \right)_+$$

$$\sum_{y'} p_{x-y'}\eta_{-n-1}^{\text{stat}}(y') \left( m - \sum_z \lambda_{z-y'}\eta_{-n-1}^{\text{stat}}(z) \right)_+$$
Ancestral lines

Given stationary \( (\eta_n^{\text{stat}}(x), n \in \mathbb{Z}, x \in \mathbb{Z}^d) \), cond. on \( \eta_0^{\text{stat}}(0) > 0 \), sample an individual from space-time origin \((0, 0)\) (uniformly)

Let \((X_n)\) position of her ancestor \(n\) generations ago:

Given \(\eta^{\text{stat}}\) and \(X_n = x, X_{n+1} = y\) w. prob.

\[
\frac{p_{x-y}\eta_{-n-1}^{\text{stat}}(y) \left( m - \sum_z \lambda_{z-y}\eta_{-n-1}^{\text{stat}}(z) \right)_+}{\sum_{y'} p_{x-y'}\eta_{-n-1}^{\text{stat}}(y') \left( m - \sum_z \lambda_{z-y'}\eta_{-n-1}^{\text{stat}}(z) \right)_+}
\]

Hopeful theorem in progress ...

If \(m \in (1, 3), 0 < \lambda_0 \ll 1, \lambda_z \ll \lambda_0\) for \(z \neq 0\), there is a regeneration construction for \((X_n)\).
Thank you for your attention!