

ON THE FOUNDATIONS OF MATHEMATICAL ECONOMICS

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Abstract:

Kumaraswamy Vela Velupillai [74] presents a constructivist perspective on the foundations of mathematical economics, praising the views of Feynman in developing path integrals and Dirac in developing the delta function. He sees their approach as consistent with the Bishop constructive mathematics and considers its view on the Bolzano-Weierstrass, Hahn-Banach, and intermediate value theorems, and then the implications of these arguments for such “crown jewels” of mathematical economics as the existence of general equilibrium and the second welfare theorem. He also relates these ideas to the weakening of certain assumptions to allow for more general results as shown by Rosser [51] in his extension of Gödel’s incompleteness theorem in his opening section. This paper considers these arguments in reverse order, moving from the matters of economics applications to the broader issue of constructivist mathematics, concluding by considering the views of Rosser on these matters, drawing both on his writings and on personal conversations with him.

Acknowledgements: I thank K. Vela Velupillai most particularly for his efforts to push me to consider these matters in the most serious manner, as well as my late father, J. Barkley Rosser [Sr.] and also his friend, the late Stephen C. Kleene, for their personal remarks on these matters to me over a long period of time. I also wish to thank Eric Bach, Ken Binmore, Herb Gintis, Jerome Keisler, Roger Koppl, David Levy, and Adrian Mathias for useful comments. The usual caveat holds.

I also wish to dedicate this to K. Vela Velupillai who inspired it with his insistence that I finally deal with the work and thought of my father, J. Barkley Rosser [Sr.], as well as Shu-Heng Chen, who supported him in this insistence. I thank both of them for this.

I. Introduction

“Beauty is truth, truth beauty.” --- John Keats, *Ode on a Grecian Urn*,
1819.

“For me, and I suppose for most mathematicians, there is another reality, which I shall call ‘mathematical reality’; and there is no agreement about the nature of mathematical reality among either mathematicians or philosophers. Some hold that it is ‘mental’ and that in some sense we construct it, others that it is outside of us and independent of us. A man who could give a convincing account of mathematical reality would have solved very many of the most difficult problems of metaphysics. If he could include physical reality in his account, he would have solved them all.”

--- G.H. Hardy, **A Mathematician’s Apology**, (1969, [30], p. 123) .

Kumaraswamy Vela Velupillai has long labored to understand the deepest underpinnings of economic theory. In earlier years this was focused on nonlinear dynamics in economics, focusing particularly on ideas drawn from his professor at Cambridge University, the late Richard M. Goodwin [67]. More recently, however, he has shifted the focus of his efforts into originating and developing the approach of *computable economics* ([68], [69], [70], [71], [72], [73], [74]).¹ In this final paper [74], “Taming the Incomputable, Reconstructing the Nonconstructive and Deciding the

¹ “Computable” economics is to be distinguished from “computational” economics, with the former focusing on fundamental issues such as the computability or undecidability of certain systems, whereas the latter tends to be more concerned with more superficial technical matters such as how to get particular programs to run more quickly. The latter also studies simulations of particular economic models.

Undecidable in Mathematical Economics,” he embarks upon an especially ambitious foray into the topic of computable economics and its deepest foundations. While different schools of constructivist mathematics accept more or less of these, the ideas that the strictest of constructivists question include the existence of actual infinities (the axiom of infinity), the law of the excluded middle, and the Axiom of Choice.²

The paper opens with a discussion of the main incompleteness theorem of Gödel, [27], and notes that Rosser [51] showed that it held for the more general *simple consistency* as well as for the more restricted ω -*consistency*.³ Recognizing the centennial of Rosser, Velupillai praises his approach for in fact involving some sacrifice of generality in order to gain a strengthening of results.

“This is the standard method that *constructive mathematicians* apply in their constant struggle to *constructivize classical mathematics*.” Velupillai, ([74], p. 4) [italics in original]

He then proceeds to praise Rosser for some of his more practical and simplifying approaches to more simple matters, in particular his authorship with Carl de Boer [54] of a guide to using pocket calculators for solving basic calculus problems, seeing in this an example of the constructivist approach that insists on providing actual algorithms for solving problems and proving theorems.

² A simple statement of this axiom due to Rosser ([52], p. 88) is “If λ is a set of nonempty, nonoverlapping sets, then there is a set γ which has exactly one member in common with each member of λ .” Given that λ can vary, which is usually a transfinite cardinal, there are many versions of the axiom, with Zorn’s Lemma [79] usually being assumed to assert it holds for most levels that most mathematicians deal with, although Specker [64] showed that it does not hold at the ultimate level of the universe as a whole, using the law of the excluded middle, or *reductio ad absurdum*, thus rendering absurd a constructivist assertion of this theorem as a general disproof of the axiom. Specker’s result uses the Quine’s New Foundations approach. ³ “Simple consistency” is Rosser’s terminology, whereas some others call this “ordinary consistency” and Gödel simply called it “consistency.” It was Gödel who introduced the concept of ω -consistency. Both of these papers, along with important ones by Alonzo Church, Alan M. Turing, Stephen C. Kleene, and Emil Post, can be found in Davis [22].

I shall return to Rosser's views on these matters in more detail in the final section of this paper, but for now let me note that he was one of those relatively rare mathematicians who fully spanned both pure and applied mathematics. While pure does not equal classical and applied does not equal constructivist, there is a perceived correlation between the two on the part of many. Thus, he would seem to span that divide in some sense, although in contrast with Hardy, whom he greatly admired, he was struck by how sometimes apparently "useless" pure mathematics, such as logic seemed in the 1930s, could end up proving to have applied uses later that were unforeseen, as the theorems about decidability and computability and recursiveness of systems more generally would come to be viewed as applied relevance after the full invention and spread of programmable computers. For Hardy [30], following John Keats, the ultimate judgment to be made on the supposedly useless (and pure) "real" mathematics should be made on aesthetic grounds.

Velupillai then argues that this tradeoff between generality and strength of argument lay at the base of the development of the Feynman path integrals [26] and the Dirac delta function [24], both very useful in physics. These ideas were criticized by mathematicians for their lack of rigor, despite their usefulness, although the search for more rigorous foundations of them would lead to developments in mathematics, such as the idea of the distributions of generalized functions due to Laurent Schwartz [60] that encompasses the Dirac delta function.

Velupillai then turns from praise to criticism, focusing on the intermediate value theorem, the Bolzano-Weierstrass theorem, and the Hahn-Banach theorem. The first is seen as usually being proved by *reductio ad absurdum* (or "proof by contradiction"), with

the strategy of assuming a particular function such as a polynomial strengthening and constructivizing it. Bolzano-Weierstrass and Hahn-Banach are both usually proven using the Axiom of Choice, which is not accepted by the constructivist approach, although Velupillai notes that the latter in particular can be proven by constructivist methods that actually show the linear functional in question.

These latter two are also seen as key to the non-constructivity of crucial “crown jewels” of mathematical economics, with Bolzano-Weierstrass deeply tied to the fixed point theorems used in the standard proofs of the existence of equilibrium in both game theory and in general equilibrium theory. Hahn-Banach plays a crucial role in the second welfare theorem of economics. The criticisms and analysis by Erret Bishop [6] play a central role in these discussions.

The problem of non-computability is then considered in the form of the *busy beaver problem* due to Radó [49]. The problems arising from this non-computable function are seen as related to the *Berry paradox*, which is also related to the *Richard paradox* ([39], pp. 36-40). Velupillai proposes “taming” the busy beaver by constraining its growth following the approach of Greenleaf [29]. In Greenleaf’s view, mathematics is reduced to being essentially a programming language, so that the old trinity of *assumption, proof, conclusion* becomes *input data, algorithm, output data*.

Velupillai concludes by noting the conventional argument by classicalists that constructivism is “too hard to learn.” He objects that most of those arguing this do not really understand the issues at hand or even all the alternatives that may exist, including *nonstandard analysis*, arguing that the conventional view is maintained largely through

inertia. He invokes Gödel on believing that the real meaning of the incompleteness results involves the invocation of “higher types” that allow for decidability.

In the succeeding sections I shall consider some of these arguments in further detail, however will largely go backwards from the economic applications to the views of Rosser (my father) on the foundations of mathematics. In so doing, we shall move from what Hardy would have deemed as “trivial” mathematics to “real” mathematics, and from the arguably more practical and applied to the purer and more metaphysical, dealing with the debate implicit in the opening quotations being our ultimate goal, even if that goal is itself ultimately elusive and undecidable.

II. Problems for Economic Theory

A. Explosive Phenomena

Velupillai draws on the work of Zambelli ([77], [78]) to consider the problem of the possible uncomputability of economic processes. The economic process Zambelli focuses on is that of endogenous growth, and the function he identifies as a metaphor for its possible uncomputability is the *busy beaver function* of Radó [49]. Velupillai ([74], Section 3.2) proposes a method of *minimalization* due to Greenleaf [29] to “tame” the otherwise uncomputable busy beaver function, although he states after making his proposal ([74], p. 39) that he is “not sure” that his “suggestion” is a “way out of, and beyond, this renewed perplexity” (of the busy beaver function). What is going on here?

The busy beaver function is defined for n -state Turing machines as the maximum number of 1's (or “productivity”) for any member of the n -state that it writes in a square on a blank, two-way infinite tape and halts, and is labeled $\Sigma(n)$, with this productivity

equaling zero if it fails to halt. This function is the largest of a finite set of non-integers. Dewdney ([23], pp. 10-11) explains the non-computability of this well-defined function as arising because it “grows too fast.” Any formula used to compute it will end up generating more 1’s for an n-state busy beaver than the formula specifies.

Velupillai sees the problem as being of a naming or semantical nature along lines similar to Chaitin ([15], [16]) and a case of the *Berry Paradox*, first identified by Bertrand Russell [57]. His classic example ([57], p. 222) takes the form of “the least integer not nameable in less than nineteen syllables.” That refers to a particular number, 111,777, but the phrase in the quotation marks itself only contains eighteen syllables, thus establishing the paradox.⁴

The solution advocated by Velupillai is to use Greenleaf’s [29] method to properly restrict the range and domain of the busy beaver function. This method was used to guarantee the primitive recursiveness of the Ackermann function, although as noted, Velupillai is uncertain that this method is fully satisfactory.

What then is the economic significance of this problem? For Zambelli it is a metaphor for the development of technological ideas in endogenous growth theory, a possible paradox that can arise from the “production of ideas by means of ideas.” The problem is in some sense that the ability to create new ideas can outrace the ability to name what is being created. On the other hand, the arbitrary nature of the halting that occurs can become a metaphor for the limits of the production of knowledge as well, which he shows in his analysis using the busy beaver function [78].

⁴ A traditional method to escape the paradoxes of Russell, including the famous one regarding whether the set of all sets that are not members of themselves is a member of itself, is to place restrictions on the use of certain names that lead to such paradoxes. It has been argued that many of the paradoxes of logic arise from problems related to “naming the names” of mathematical objects.

It occurs to me that another area of economics where the questionable computability of the busy beaver function might arise might be is with the problem of explosive hyperinflation, where the very act of agents attempting adjust their expectations to the ever-accelerating rate of inflation itself triggers a further acceleration that pushes it ever beyond the ability of their expectational formulations to compute. An aspect of this that is rarely commented upon has a certain connection with the famous remark by Keynes ([38], Chap. 12) regarding the beauty contest regarding how agents may begin to think in terms of higher order expectations, not merely guessing the average guess of the other contestants, but the average guess of the others about the average guess of the others, and higher. However, in this case it may be a matter of focusing on higher order derivatives without limit. As the hyperinflation accelerates agents may cease forecasting the rate of inflation but instead focusing on the rate of change of the rate of inflation and then the rate of change of the rate of change, and so forth, with these rising expectational dynamics themselves driving the system to accelerate ever ahead of these evolving expectations.

Yet another possible example is that implied in the markomata argument of Mirowski [43]. While he limits his discussion to the four level hierarchy of Chomsky [17], in principle there is no reason that there might not be more levels, with no clear upper bound. This is a model in which simpler markets generate higher order markets that embed the lower level ones, much as a futures market may embed a spot market, and an options market may then embed a futures market. We may have seen the outcome of such a process in the financial collapses 2007 and 2008 as ever higher order derivatives were created out of lower order ones in a way that kept the system from achieving a

computable general equilibrium solution, although this must be admitted to be a rather speculative possible application of these ideas, with such a process not clearly related to the busy beaver or any other clearly uncomputable function.

Of course we should keep in mind that in the applications by Zambelli the ultimate process being modeled is not explosive, even if the function that is being used to model them is.

B. The Second Welfare Theorem and the Hahn-Banach Theorem

Velupillai ([74], Section 3.1.2) summarizes the presentation of both the Hahn-Banach Theorem and its use in proving the Second Welfare Theorem of Economics as carried out by Lucas, Stokey, and Prescott [42]. This offers an example of the contrast between a classical and a constructive approach to a particular economic problem. The Second Welfare Theorem shows that for suitable convexity conditions on sets and continuity conditions on functions, then for any Pareto optimal allocation there is a Walrasian equilibrium price vector that can support it. Use of an appropriate separating hyperplane is part of most of the proofs of this theorem to guarantee the existence of the appropriate price vector, with the Hahn-Banach Theorem generally invoked for this part.

The classical version of the Hahn-Banach Theorem involves a degree of precision that is not supplied by the constructivist approach, but that is in turn dependent on assuming a strong version of the Axiom of Choice to avoid the problem of algorithmic determination. In its classical form the Hahn-Banach Theorem allows the extension of a bounded linear functional from a linear subset of a separable normed linear space to a functional on the whole space with identical norm, which will provide the price vector in the Second Welfare Theorem. However, Velupillai notes that a constructive proof of the

Hahn-Banach Theorem introduces an element of approximation, only allowing for the differences in the norms of the two functionals to be less than some finite $\varepsilon > 0$, as discussed by Nerode, Metakides, and Constable [45] and further developed by Ishihara [34]. It only holds for subspaces of separable normed spaces. The key difference is that while the classical approach asserts an exact solution, while not providing it, the constructive approach provides an actual solution, which is, however, not the exact solution asserted in the classical approach, although arbitrarily close to it.⁵

A rather curious possibility that also arises from the Hahn-Banach Theorem has been pointed out by Velupillai ([74], p. 16, footnote 20). This is from a result due to Pawlikowski [48] that the Hahn-Banach Theorem implies the Banach-Tarski paradox. This leads Velupillai to declare “tongue in cheek” that he is tempted to show that the Banach-Tarski paradox is implied by the Second Welfare Theorem, which would destroy the idea that “there is no such thing as a free lunch.” The Banach-Tarski paradox shows that a ball can be cut up in certain ways and then reassembled after appropriate rotations and translations to form a larger ball [4], which has sometimes somewhat humorously been described as the idea that a pea could be cut up and reassembled to form the sun. Such an operation would make the busy beaver look like he is asleep. However, it is understood that this is one of those theorems that only holds in “mathematical reality” and not in physical reality, in contrast with the serious reality that spacetime obeys non-Euclidean geometry, which was originally conceived of as an odd mathematical trick.

⁵ Velupillai also notes that there is a great similarity between the constructive approach to proving the Hahn-Banach Theorem and the Intermediate Value Theorem, which in turn has indirect links with the Brouwer Fixed Point Theorem. However, we shall not consider Velupillai’s discussion of the intermediate value theorem further in this paper, as it is not tied to any specific economic application, nor does it involve any further mathematical concepts beyond what we deal with in the Hahn-Banach and Bolzano-Weierstrass theorems. I thank Eric Bach for pointing out to me that the finite-dimensional version of the Hahn-Banach Theorem can be proven without any recourse to the Axiom of Choice or its weaker ultrafilter relatives.

This is because the Banach-Tarski paradox involves cutting the ball into a countably infinite set of subsets.⁶

C. The Existence of Equilibrium

There can be little question that the ultimate “crown jewel” of economic theory consists of the proofs of the existence of economic equilibrium, whether game theoretic or Walrasian general competitive. Central to these has been the use of various fixed-point theorems, with the first to be used being the original one, that due to L.E.J. Brouwer [9] and the second being its close cousin, that due to Kakutani [35]. In both cases the application was first made in game theory and then later to general Walrasian equilibrium. It was von Neumann [46] who initiated this exercise in proving the existence of minimax equilibria for mixed strategies in certain games. He would follow this by using it again to provide the first proof of the existence of a competitive equilibrium [47], although one in the rather particular form of a balanced-growth path. Following in the path of von Neumann, it would be John Nash [44] who would first apply the Kakutani variant [35] to prove the existence of equilibrium for non-cooperative games, which inspired Arrow and Debreu to use it in their proof of the existence of Walrasian general equilibrium [2].

While Velupillai discusses contraction theorems and the Schauder fixed point theorem, most of his focus is on the foundational Brouwer fixed point theorem. I shall focus my intention on it, especially given the contradictions and issues arising from Brouwer’s role in inventing intuitionist mathematics [8], which was in conflict with the

⁶ The pieces of a Banach-Tarski decomposition must be *non-measurable*; therefore by the celebrated result of Robert Solovay [63] such decompositions cannot be built without some use of the Axiom of Choice and hence lie outside constructive mathematics, suggesting that perhaps Velupillai is right to put his tongue into his cheek when referring to them. (I thank Adrian Mathias for clarifying this matter for me.)

methods he used (and those used by others since) to prove his most celebrated theorem.⁷ Velupillai discusses the generally non-constructive nature of two different approaches to proving Brouwer's fixed point theorem.

One that is used most frequently in mathematical economics (e.g. Scarf [58]) relies fundamentally upon the Bolzano-Weierstrass theorem that essentially states that every bounded sequence contains a convergent sequence.⁸ Ultimately these proofs end up selecting such a convergent sequence that conveniently ends up approaching the fixed point. However, this involves invocation of strong versions of the Axiom of Choice to find this sequence.⁹ To add further to the problems with the Bolzano-Weierstrass theorem from the constructivist perspective, especially its intuitionist variant, the theorem also relies on proof by *reductio ad absurdum*, that is the law of the excluded middle, the very idea the rejection of which lies at the heart of the intuitionist philosophy {Dummett [25]}).

The other main approach to proving the theorem is that of Brouwer himself [9]. As noted by Velupillai, Brouwer went at this in a highly indirect way that ultimately relied upon the logical equivalence between a proposition and its contrapositive and the law of double negation. So, first Brouwer showed that given a map of the disk onto itself with no fixed points there exists a continuous retraction of the disk to its boundary. Then he showed its contrapositive that if there is no continuous retraction of the disk to its boundary then there is no continuous map of the disk to itself without a fixed point.

⁷ Brouwer [10] did eventually follow up to provide a correction to his proof to make it compatible with intuitionism.

⁸ Often it is Sperner's lemma that is invoked, but this depends on the Bolzano-Weierstrass theorem for its proof. See Tompkins [66] for further discussion.

⁹ Needless to say, there are economists who find showing how difficult it is to prove convergence to a fixed point to be quite uninteresting in the face of experimental evidence that at least in experimental double auction markets most agents are able to move to (partial) equilibrium solutions quite rapidly with little thought [62].

When Brouwer developed intuitionism he fully understood how this proof did not correspond with it.

Brouwer's intuitionism involved a strong form of constructivism very much in the tradition of Kronecker who had debated with Cantor in the 19th century. While the law of the excluded middle was to be disavowed, the Axiom of Infinity was also denied, although potential infinity was allowed (that there is no upper limit to the natural numbers, even if there is no meaning to their aggregate constituting an infinite set). Many modern constructivists are more willing to allow the Axiom of Infinity, but then draw the line at higher levels or logics based on the existence of higher levels of infinity, the proof of whose existence by Cantor involved reduction ad absurdum. However, in contrast to the computable emphasis of many constructivists, Brouwer's concerns were ultimately more philosophical and even mystical [11]. His emphasis on possibly allowing for something to be both true and false can lead to the sort of transcendental perspective that is often argued to arise upon contemplating the apparently absurd Zen koans,¹⁰ (as well as possibly Marxist dialectics) even as it stands aside from the usual conflict between Aristotelianism (which asserts the Law of the Excluded Middle) and Platonism (which is rejected by all the constructivists).

However, some theories thought to be related to intuitionism may simply involve probabilistic statements regarding degrees of truth, which may themselves still be

¹⁰ While most western works on Zen Buddhism emphasize this view of the koans, there is another more "classical" approach in which they have specific answers that the adept is expected to learn and repeat in a nearly rote way, once learned. Hoffman [33] provides a "cheat sheet" of answers for some of the more famous ones, with, for example, the official answer for the most famous one of all, "what is the sound of one hand clapping?" being simply the act by the adept to stand correctly before the master and to thrust his hand forward decisively. Thinking of this sort of Zen leads to understanding how it could be related to an authoritarian use of martial arts.

declared to be true or false, as in fuzzy set theory [76]¹¹ or Boolean algebra [13]. In any case, while many view intuitionism as philosophically attractive, the formalism of Hilbert would dominate the leading mathematics journals from the 1920s on, leaving Brouwer and his main follower, Heyting [32], somewhat isolated until Kleene [39] and then Dummett [25] would come to their defense.

III. Can the Struggle within Mathematics be Resolved?

Much as economics is riven with vigorous debates between different schools of thought, so it is the case within mathematics, even if there is a dominant school that is often labeled “classical.” Much as heterodox economics schools struggle with and against the dominance of neoclassical orthodoxy within economics, so do the constructivists struggle against this dominant classical approach within mathematics, even as this oversimplifies the lineup and classification of schools within mathematics. Indeed, within logic, the classical approach is usually thought to contain two schools, the “logicistic” of Russell and Whitehead and the “formalistic” of Hilbert and von Neumann, with the intuitionists being the third school that is more on the outside within the broader constructivist camp ([39], pp. 46-53). However, within mathematics more broadly there are other schools as well, just as the division between “heterodox” and “orthodox” within economics is a drastic oversimplification [20].

Within this major paper [74], Velupillai makes the case for constructivism by appealing to certain mathematical ideas invented by physicists that were looked down

¹¹ This is an interpretation that Gödel [28] appears to have accepted. However, others distinguish more sharply between “many-valued logics” that assign definite truth values (interpretable as probabilities of truth), and intuitionism, which is seen as less definite on such truth values by Kleene [39] as well as Rosser and Turquette [56]. More recently, Atanassov [3] has proposed a combination of the approaches in the theory of “intuitionistic fuzzy sets.”

upon by rigorous, classical mathematicians,¹² but which in the end have come to be accepted, and were used by physicists and engineers before that happened because of their practicality. One is the idea of path integrals developed by Feynman [26] and the other is the Dirac delta function [24]. Of these two, Feynman was more concerned with the practicalities involved rather than the philosophical or strictly mathematical implications compared to Dirac, who developed quite strong views about these latter matters. However, for Velupillai the great appeal of both of them is their willingness to accept approximations that work rather than being obsessed with an unattainable idea of idealistic perfectionism.

Feynman's sin was to solve for quantum mechanics an equation by simultaneously carrying out integrals over all space variables at each point in time, giving the "integral over all paths," a concept that remains to this day non-axiomatized, with some labeling it as "mathematically meaningless" if still "impressive for the geniality of the physical intuition underlying it" and "one of the greatest achievements of 20th – century's theoretical physics" ([61], p. 3). Velupillai ([74], p. 9) argues that it is best interpreted as being a "rule of thumb algorithm." This certainly places it into the constructivist tradition, even if Feynman himself cared little about such debates.

Dirac's delta function is perhaps even more troublesome from the standpoint of much of classical mathematics, derived as it is from the earlier (and also intuitive) Heaviside function (which is the integral of the Dirac function). So, the Heaviside function is a flat function except for a discrete step at one point, while the Dirac delta function is zero everywhere except for at the point of the step in the Heaviside function, where it is infinite. It does not look like a meaningful function at all, and it would be

¹² The classical and rigorous approach to mathematical physics is codified in Courant and Hilbert [21].

criticized by the likes of von Neumann [41] until it was rescued from non-respectability by Laurent Schwartz [60] inventing the concept of generalized functions and their distributions within which the delta function could fit. Again, like Feynman after him and Oliver Heaviside before him, Dirac was driven by intuitions about physics rather than mathematical rigor. Dirac was also motivated by aesthetics, preferring ideas that seemed possessed of beauty, thus perhaps agreeing with Hardy in being a follower of John Keats and his *Ode on a Grecian Urn*.

Curiously, besides interpreting the Dirac delta function as being generalized, yet another school of mathematical thought allows for the possibility of its being an exact and well-defined function, namely *nonstandard analysis*, with such a possibility being established by Todorov [65]. Developed by Abraham Robinson [50], nonstandard analysis looks on the surface to be the exact opposite of constructivist mathematics, and many consider this to be the case, with Errett Bishop [7] leading a charge specifically against using it in the teaching of elementary calculus as proposed by Keisler ([36], [37]). Bishop argued forcefully that it involved a debasement of numerical meaning, as well as an excessive reliance on the use of the Axiom of Choice in its development by Robinson.

Another problem is that whereas constructivism in its intuitionist formulation eschews the infinite, nonstandard analysis positively glories in it. Not only are the transfinite cardinals accepted, but an entirely different set of infinite aggregates are allowed, *hyperreal* numbers that are infinite. The real payoff for allowing these is that their reciprocals are infinitesimals, numbers arbitrarily close to zero, but not equal to it. Indeed, Robinson did not invent this idea, rather he resurrected it, arguing reasonably that when Leibniz originally developed his version of the calculus, he conceived of

derivatives as ratios of infinitesimals, rather than as limits of sequences of finite numbers. His great rival, Newton, apparently used both ideas in his development of “fluxions,” with Robinson ([50], p. 280) suggesting that Newton was reluctant to admit to infinitesimals in the face of attack from Berkeley who denied the existence of actual infinities.¹³ However, Robinson ([50], p. 281-282) notes that it is possible to use infinitesimals, which certainly appeal to the intuition of many introductory calculus students, without necessarily believing that they really exist, saying that “the intuitionists and other constructivists” might agree with Leibniz who declared regarding infinitely small and large numbers, “que ce n’étaient que des fictions, mais des fictions utiles.” He then argues that while the “majority of mathematicians” agree with the “platonistic” classicism of Cantor that infinities are ontologically real, for a logical positivist the question is meaningless, even while one “would concede the historical importance of expressions involving the term ‘infinity’ and of the (possibly, subjective) ideas associated with such terms.”

Now it would appear in this paper by Velupillai [74] that he recognizes nonstandard analysis as a possible alternative to constructivism and computable approaches, but he does not provide any indication of any enthusiasm for it. This may reflect how it has been used in the past in economics, largely for proving minor extensions of standard proofs of existence of general equilibrium, allowing for conceptually larger numbers of agents, but still based on such non-constructive concepts as the Axiom of Choice ([13], [12], [1]). Nevertheless, in his paper Velupillai cites some

¹³ I thank David Levy for pointing out to me that almost certainly whatever ambiguities Newton may have entertained regarding the nature of his fluxions, they were not due to Bishop Berkeley’s jibes against them as the “ghosts of departed quantities,” as Berkeley did not make his arguments until 1734 in his *The Analyst*, whereas Newton died in 1727.

items that include some efforts to reconcile constructivism and nonstandard analysis, notably Schechter [59].

Indeed, that effort has been a gradually building enterprise, perhaps initiated by Wattenberg [75], who noted that the effort at reconciliation was coming more from advocates of nonstandard analysis than from the harder line constructivists such as Bishop. While Robinson and Keisler developed it to be essentially an extension of classical formalism, not contradicting the conventional view in any way, the newer efforts make this effort to be more consistent with constructivism, especially in its intuitionist formulation. Thus, Geoffrey Hellman [31] points out that the essential argument about infinitesimals involves the very intuitionistic idea that they are both equal to zero while also not being equal to zero. In the end, Hellman argues for a “pluralism” of mathematical systems, in this regard reflecting the arguments of some supporters of more heterodox approaches to economic theory.

IV. The Position of J. Barkley Rosser [Sr.]

“The main disadvantage of a system of symbolic logic is that it is a formal system divorced from intuition.” --- J. Barkley Rosser, **Logic**, (1978, [53], p. 10) .

“One advantage of a symbolic logic is that it can be made very precise, but an even greater advantage is that it can be changed to fit the circumstances.”

--- J. Barkley Rosser, **Logic**, (1978, [53], p. 522).

As promised at the beginning of this paper, we now arrive at the discussion Velupillai started his paper with, a consideration of the views of my late father, J. Barkley

Rosser [Sr.], on these matters as best I understand them. based both on reading his work and remembering discussions with him (he died in 1989), with some of the more revealing discussions involving his close friend and long associate, Stephen C. Kleene as well [40]¹⁴. In this discussion we shall deal with the deeper foundational issues..

Velupillai suggests that Rosser was an ally of the constructivist position, and I have no disagreement whatsoever with his interpretation of the extension of the incompleteness theorem of Gödel made by Rosser in 1936 [51] as supporting the “classic trade-off that great mathematicians consciously make between weakening/strengthening conclusions and weakening/strengthening hypotheses” ([74]. p. 4). Furthermore, although as time proceeds it is for his earlier more pure, logic work that he is mostly remembered, Rosser was a serious applied mathematician later on, indeed always had this orientation to some degree as he had a masters degree in physics before he earned his Ph.D. in mathematics at Princeton under Alonzo Church.¹⁵ That background would manifest itself during World War II when he was perhaps the leading expert on rocket ballistics in the U.S.,¹⁶ and his first book was *The Mathematical Theory of Rocket Flight* [55] (still in print), which reflected his deep fascination with the exploration of space by humans,¹⁷ later shown by

¹⁴ In this joint paper from 1935 [40], they showed that the λ -calculus is inconsistent, although modified versions of it have since been proven to be consistent. They were among the developers of this system, much used later by computer scientists and artificial intelligence programmers. Others closely involved in its initial development were Haskell Curry and Alonzo Church, the latter the major professor of both Kleene and Rosser.

¹⁵ It may be that now the most intensively studied of his theorems is the one he proved with Church that shows the “diamond property” of recursive systems, the Church-Rosser theorem [18], which was very pure mathematics when they proved it prior to the invention of computers with memory, but is now viewed as deeply practical by computer scientists.

¹⁶ Like his friend, John von Neumann, he continued to be involved with the military and intelligence after that war, receiving numerous commendations for his mostly classified work, including one for his solution of the water-to-air phase transition problem of the first submarine-launched missile, the Polaris. Later, during the Vietnam War this work of his would become a matter of public controversy [5].

¹⁷ In 1929 he forecast to a group of highly skeptical friends that humans would land on the moon within 50 years. When this event occurred in 1969, he said that what he had not forecast was that he would be able to view it live on television (which he rarely watched otherwise, mostly considering it a “waste of time”).

his deep involvement in the space program of the U.S.¹⁸ Furthermore, this deep pragmatism as reflected in the quotation above regarding the importance of being able to “be changed to fit the circumstances” suggests that he may have been open to any viewpoint as long as it could satisfy this ultimate criterion.

Without question he was a deep student of the controversies regarding the nature of the foundations of mathematics. His acute awareness of these debates is given by the following summary of them by him regarding the Axiom of Choice.

“X’s position: Any finite number of choices is permissible, but not an infinite number.

Y’s position: A denumerable number of choices is permissible, but not for any larger number.

Z’s position: Any number of choices is permissible.

Among actual mathematicians, perhaps a majority would applaud X’s position, certainly some would agree with Y, and others (who perhaps constitute a minority) agree wholeheartedly with Z. However, many mathematicians who would like to espouse the positions of X or Y find that this would leave them with no means of proof for certain theorems which they need in their research.

Accordingly, they accept the position of Z, but with reluctance and a hope that someone will one day find proofs of their key theorems which do not involve an infinity of choices. Thus it comes about that we find papers written in which all

¹⁸ He played a crucial role in the first successful manned flight to the moon. He solved a problem of astronauts landing on earth further from their planned rendezvous sites with longer flights as due to a discrepancy between the clocks on the ground and those in space, with the former on solar time and the latter on sidereal (star) time, off from each other by the roughly 1/365 due to the annual revolution of the earth about the sun. If this had not been resolved, the first flight to the moon would have gone into deep space without hope of return.

the results of the paper depend on a theorem whose only known proof involves an infinity of choices; nevertheless, throughout the paper the author is careful to avoid the use of an infinity of choices.” --- Rosser, (**Logic**, 1978 [53], p. 491).

Velupillai and his allies in the program to constructivize the foundations of mathematical economics have argued precisely this last point at some length, that mathematical economist after mathematical economist has sought to prove the existence of general equilibrium and other crown jewels of economic theory, while assiduously avoiding mentioning the fact that their proofs are relying on unmentioned axioms that are not at all agreed upon universally. In this regard Rosser certainly had sympathy with these constructivist arguments, and his extensive research in numerical analysis and computer science more generally reinforces this perception.

However, we are now coming to the more difficult point of ascertaining his real position, and here I shall report my own personal observations of him. Something that makes things difficult is that while he often had strong views regarding things, he also tended to avoid specifically declaring his position, indeed enjoyed not doing so. To make matters worse, he would sometimes play the “devil’s advocate” and brilliantly argue for a position he disagreed with just to provoke his interlocutors.¹⁹ In the area of politics he strongly asserted the right of the citizen to secrecy of the ballot, and I never heard him actually state whom he had voted for in any election, although I generally had a good idea whom he favored.

¹⁹ While he often enjoyed confounding others, he could also be reassuring when it suited him. Thus, during a public presentation, a young woman asked him if zero was a real number. Ever the perfect southern gentleman, he replied, “One of the finest, my dear, one of the finest.”

There is a non-trivial relationship between these attitudes and the implications of the Gödel theorems with which he was so deeply associated. So within a system, both a statement and its opposite may be able to be proven. One response to this might well be intuitionism: overturning the law of the excluded middle can allow us to assert that both are true. However, another view, implied by the quote by Gödel at the end of Velupillai's paper, is that there may be a definitely correct answer, a definite truth, but that it can only be known by moving to a "higher type," that at a higher level of perspective the answer can be known. It was his hard pragmatism that inclines me to the view that he was not a post-modernist with regard to reality: he had a very hard appreciation of the very hard reality of the very hard facts that make themselves unavoidable to us, even if there are higher order things that we do not know or perceive whose reality or lack thereof remains a mystery.

So, while he never expressed his bottom line opinion on these matters, I have a sense of where he stood on some of them, especially compared to his old and close friend, the late Steve Kleene. I think that Kleene was more the constructivist, indeed even an intuitionist, whereas my father was in his heart of hearts a classical mathematician who accepted the law of the excluded middle, and who agreed with Hardy that mathematical reality truly exists outside of us. Let me deal with these issues in order.

Regarding the law of the excluded middle, one piece of evidence, exhibited throughout his writings, is that he had no hesitation in using *reductio ad absurdum* arguments. I think that he viewed it as a matter of beauty implying truth, with a good proof by contradiction being the highest form of mathematical aestheticism. When I was

13, he sat me down one day and worked through the proof that there is no highest level of infinity. This is a diagonal proof by contradiction in which one assumes that one has found a set that possess the highest possible level of transfinite cardinality, and then one examines its power set, the set of all its subsets, and shows that one cannot establish a one-to-one correspondence between this set and the one that was assumed to be at the highest level of infinity, with the failure of the one-to-one correspondence amounting to there always being at least one member of the power set that is left over not being associated with a member of the original set, just as in Cantor's proof that the real number continuum is at a higher level of infinity than the denumerable natural numbers.

In this he contrasted with his friend Kleene. At a certain point I became fascinated by arguments regarding Zen koans and the possible reality of apparent contradictions holding simultaneously. When I asked my father about this he told me to ask Steve Kleene about it. When I did, Kleene told me about intuitionism, the first time I heard about it. Now, while my father makes frequent references to "intuition" in his works, he never used the term "intuitionism" that I am aware of. I think that in the end he simply did not buy it, although I suspect that it also involved the view of Keats, and that it was fundamentally an aesthetic judgment: he found a good proof by contradiction to be supremely elegant and beautiful, and intuitionism rules them out.

The question reappears implicitly in his 1969 book, *Simplified Independence Proofs* [52]. There he redoes the forcing proof by Paul Cohen [19] of the independence of the continuum hypothesis into Boolean algebraic terms. As noted earlier, like fuzzy logic, Boolean algebra admits of truth values that are not just one or zero. This would follow the approach developed by Rosser and Turquette [56] in their work on multiple-

valued logic. This looks on the surface like it could be consistent with the possible intuitionistic assertion that A may be both true and not true at the same time. But this is not the case in the original formulations. The Boolean formulation can sometimes be interpreted as asserting probabilistic truth, as in the many-valued logic.²⁰ Thus the statement that “there is a 30 percent chance of rain today” can be viewed as an Aristotelian assertion: it may be definitely true or false (although it may not be easy to establish if it is or not, as the appearance of rain today does not disprove it). Booleanism and Aristotelianism are perfectly consistent, if considered within the penumbra of many-valued logic.²¹

Another piece of this is the matter of whether or not one “invents” or “discovers” theorems. Are theorems “constructed” (invented), or are they expositions about a real mathematical reality outside of us that the mathematician discovers? I can report that my father always used the latter terminology: for him, for all his willingness to consider many different possible systems of logic and mathematics, theorems were “discovered.” They are sitting out there somewhere for the brilliant mathematician to find them. Thus, while he never expressed a per se opinion on the matter, I think that he ultimately agreed with Hardy that mathematical reality is a real Platonic reality.²²

²⁰ Adrian Mathias has pointed out to me that more properly speaking the Boolean valued model approach to forcing assigns truth values in a complete Boolean algebra to sentences of the forcing language. This may have nothing to do with probabilities of truthfulness per se.

²¹ However, Atanassov [3] has more recently developed an intuitionistic extension of fuzzy set logic. See also [14].

²² A sideshow to this discussion involves theology. Does believing in an ideal, Platonic, mathematical reality imply that one must believe in a deity? Hardy is proof that this is not necessarily the case, as he was both a publicly confirmed and vigorous atheist, while also accepting that mathematical reality is a Platonic ideal outside of us. As for my father’s theological views, I will note that while he attended church regularly with my very religious mother, he never definitively stated his position on these matters in my hearing, and I am less certain of his ultimate position regarding them than I am about what his political views were.

Yet another piece of evidence is that he was quite an admirer of nonstandard analysis, supporting the tenuring of Robinson's strong ally, H. Jerome Keisler, in the mathematics department at the University of Wisconsin-Madison (Kleene also was involved as well, initially hiring Keisler when he was department chairman,). In the second edition of his *Logic for Mathematicians* (which was simply entitled, *Logic* for the second edition), he added several appendices [53]. The final one (D) is about nonstandard analysis and is quite sympathetic to its arguments. In the next to last paragraph of the entire book ([53], p. 560), he cites its ability to provide an easy solution for the Dirac delta function, and in the final paragraph he approvingly cites Keisler's book, *Elementary Calculus* [36] which Errett Bishop had strongly criticized [7]. Of course, as noted above, Hellman and others see nonstandard analysis as ultimately consistent with a constructivist approach,²³ and Abraham Robinson himself fell back on a pragmatic approach, ultimately arguing that the ontological status of infinitesimals was meaningless, even as they might be useful, if only for understanding the history of mathematics.

However, in the end, I think that for my father it all came together. I believe that he agreed with "Z" above from his *Logic*, but like Hellman he saw a consistency between the pragmatic and computable aspect of constructivism in its broader perspective with there being a real existence of a higher mathematical reality that includes infinitesimals and transfinite cardinals, even inaccessible ones. More than once I heard him repudiate Hardy's claim that "real" mathematics was ultimately "useless," noting that theorems that he proved when he was young and thought to be strictly part of totally pure mathematics,

²³ I thank Jerome Keisler for informing me that the understanding of this consistency and the tradeoffs involved has now proceeded very far.

turned out later to have practical applications. Thus, I think he believed in the ultimate unification of the Platonic ideal with the Aristotelian material reality somehow. If there was a deity for him, that being was somehow connected with that level of inaccessible transinfinity that he seemed to be trying to reveal to me without saying it when he showed me the proof that there is no highest level of infinity, when I reached the age that a young man is supposed to decide his religious identity, but which he may have seen as also manifesting itself somehow in the most concrete reality that we live in.

V. Conclusions

Kumaraswamy Vela Velupillai has provided us with a stunning tour de force in his paper [74] that shows how one may move to redefine the foundations of mathematical economics upon constructivist principles. He successfully invokes arguments of J. Barkley Rosser [Sr.] in this effort and highlights the crucial theorems upon which the “crown jewels” of economic theory rest that depend upon non-constructivist arguments. He takes us through the arguments regarding mathematics and the relevant theorems, finally bringing us to the economics itself. This is quite an achievement. It may be that the person he invoked would argue that he should extend this approach to include a constructivist application of nonstandard analysis as well, but he hints at this possibility. To honor Vela’s commitment to the reconstruction of the foundations of economic theory upon a constructivist methodology, I shall conclude this paper by quoting a poem that I believe is consistent with this view.

“She was the single artificer of the world

In which she sang. And when she sang, the sea,
 Whatever self it had, became the self
 That was her song, for she was the maker. Then we,
 As we beheld her striding there alone,
 Knew that there never was a world for her
 Except the one she sang, and singing, made.”
 --- Wallace Stevens, *The Idea of Order at Key West*, 1934.

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