SELECTED APPLICATIONS OF LOGIC TO C*-ALGEBRAS

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During the last decade the interface between set theory and operator algebras advanced from virtually nonexistent to a lively area of research. I will focus my attention to two rather different topics in this area.

Concrete C*-algebra is a norm-closed, self-adjoint algebra of bounded operators on a complex Hilbert space. An abstract C*-algebra is a Banach algebra with involution that is isomorphic to a concrete C*-algebra. Essentially by a 1942 result of Gelfand–Naimark–Segal, C*-algebras are axiomatizable in the logic of metric structures ([1], see [10]). The category of abelian C*-algebras is equivalent to the category of locally compact Hausdorff spaces, and therefore the study of C*-algebras is sometimes called ‘noncommutative’ or ‘quantized’ topology (cf. [3]).

0.1. Classification of nuclear, separable C*-algebras. Elliott’s program of classification of separable nuclear C*-algebras by K-theoretic invariants has enjoyed tremendous success and achieved a number of spectacular results ([19]). However, counterexamples constructed by Rørdam and Toms have shown that the program in its original formulation needs to be revised (see [6]).

Set-theoretic analysis of the Elliott program was initiated in [12] and [11] as a continuation of the broad endeavour to study the comparative complexity of classification problems (see [15] and [14]). I will present the current (rapidly changing!) state of the art on this subject.

0.2. Rigidity of corona algebras. The multiplier algebra \( M(A) \) of a C*-algebra \( A \) is the non-commutative analogue of the Čech–Stone compactification and it is the maximal C*-algebra that has \( A \) as an essential ideal. The quotient \( M(A)/A \) is called corona algebra (or outer multiplier algebra) and is a noncommutative analogue of the Čech–Stone remainder of a topological space. Calkin algebra is the simplest non-commutative corona and the simplest non-commutative instance of the rigidity problem for corona algebras is the 1977 question of Brown, Douglas and Fillmore whether Calkin algebra has outer automorphisms. By [18] and [8], the answer to this question is independent from ZFC. I will present known results and open problems on this question (see [9], [4]).

0.3. References. Only the basic knowledge of descriptive set theory and functional analysis will be assumed. Prerequisites for §0.1 and §0.2 roughly correspond to the prerequisites for [12] and [8], respectively. I shall nevertheless list some suggested readings. For the general theory of C*-algebras see [2], [17], or [5]. For their set-theoretic aspects see [20] and [13]. An excellent reference for descriptive set theory is [16], see also [14]. In §0.2 I will use model theory of metric structures ([1], [10]), combinatorial set theory, and what not (see [8], [7]).
References


