On Potential Games and Generalized Nash Equilibrium Problems

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Outline

1 Motivation and Background

2 A Decentralized Algorithm for Solving Potential GNEPs

3 Numerical Studies

4 Conclusion and Future Work
Motivation: Decentralized Decision Making

- Unmanned air vehicles (UAVs) group coordination
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Question: Can dynamic interaction lead to a Nash equilibrium?
Motivation: Difficulties in Solving GNEPs

GNEP $\leftrightarrow$ QVI $\Rightarrow$ VI

lost in translation

Questions:

- Is there empirical or experiment evidence to support common-multiplier solutions?
Motivation: Difficulties in Solving GNEPs

GNEP $\iff$ QVI $\implies$ VI

*lost in translation*

Questions:

- Is there empirical or experiment evidence to support common-multiplier solutions?
- Can GNEPs be solved in a decentralized fashion with provable convergence?
Potential Games: [Monderer & Shapley 96]

Let $G = (\mathcal{F}, X = \prod_{f \in \mathcal{F}} X_f, (\theta_f))$ represent a strategic-form game.

**Ordinal Potential Games**

A function $\Phi : X \rightarrow \mathbb{R}$ is called an ordinal (exact) potential function for the game $G$ if for each $f \in \mathcal{F}$ and all $x_f \in X_f$,

$$\theta_i(y, x_{-f}) - \theta_i(z, x_{-f}) > 0 \iff \Phi(y, x_{-f}) - \Phi(z, x_{-f}) > 0, \forall y, z \in X_i.$$
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**Significance of Potential Games**

Converts a Nash equilibrium problem to a SINGLE optimization problem

$$\min_x \Phi(x) \quad \text{subject to} \quad x \in X.$$  

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Caveat: A Nash equilibrium of $G$ $\iff$ Global optimizer of $(P)$. 

Known Classes of Potential Games

- Congestion games [Rosenthal 73].
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- $\theta_f(x_f, x_{\neg f}) = \theta_f(x_f)$ (but $X_f = X_f(x_{\neg f})$)
  
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Workshop on Complementarity And Its Extensions, Singapore, 2012
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- \( \theta_f(x_f, x_{-f}) = C(x_1, , x_F) + d_f(x_f) \)
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Benefits of Studying Potential GNEPs

- Study the *evolution* of games (best-response dynamics)
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- Provide a **focal point** among **multiple equilibria**
- Amenable to **decentralized** control/decision-making
- Easier to study the effects of **bounded rationality** (better-response vs. best-response)
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Game Formulation and Assumptions

Generalized Nash Equilibrium Problems (GNEPs)

Each agent $f$ solves the following problem, parameterized by $x_{-f}$,

$$\begin{align*}
    \text{minimize} & \quad \theta_f(x_f, x_{-f}) \\
    \text{subject to} & \quad x_f \in X_f(x_{-f}).
\end{align*}$$

(GNEP)

GNE: $\theta_f(x^*_f, x^*_{-f}) \leq \theta_f(x_f, x^*_{-f}), \quad \forall x_f \in X_f(x^*_{-f})$. 

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\[\theta_f(x_f^*, x_{-f}^*) \leq \theta_f(x_f, x_{-f}^*), \quad \forall x_f \in X_f(x_{-f}^*).\]
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#### Assumptions

\[\theta_f(x_f, x_{-f}) : \mathbb{R}^n \rightarrow \mathbb{R}, \text{continuously differentiable and convex in } x_f\]

Non-shared constraints:

\[X_f(x_{-f}) := \{ x_f \in \mathbb{R}^n | g_{fl}(x_f, x_{-f}) \leq 0 \} \]

\[g_{fl}(\cdot, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m_f \text{ with } g_{fl}(\cdot, x_{-f}) \text{ continuously differentiable and } g_{fl}(\cdot, x_{-f}) \text{ convex}, \ l = 1, 2, \ldots, m_f.\]
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- $g_f(x_f, x_{-f}) : \mathbb{R}^n \to \mathbb{R}^{m_f}$ with $g_{fl}(\cdot, \cdot)$ continuously differentiable and $g_{fl}(\cdot, x_{-f})$ convex, $l = 1, 2, \ldots, m_f$. 
Summary of Results in the Literature

- Naive Gauss-Seidel doesn’t work even for potential GNEPs [Facchinei et al. 11]
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Our idea to solve potential GNEPs with non-shared constraints: Regularization (Facchinei & Kanzow 10) + Exact penalty (Facchinei et al. 11)
Hybrid Gauss-Seidel with Exact Penalty Algorithm

Let $g^+(x_f, x_{-f}) := \max\{0, g(x_f, x_{-f})\}$. Each agent $f$ solves

$$\min_{x_f} P(x_f, x_{-f}; \rho_f) := \theta_f(x_f, x_{-f}) + \rho_f||g^+_f(x_f, x_{-f})||_\gamma,$$

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Key Steps in the Algorithm

At a Gauss-Seidel iteration $k$
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At a Gauss-Seidel iteration \( k \)

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\text{for } f = 1, \ldots, F \text{ do compute a solution } x_f^{k+1} \text{ of}
\]

\[
\min_{x_f} \theta(x_1^{k+1}, \ldots, x_f, x_{f+1}^k, \ldots, x_F^k) + \tau ||x_f - x_f^k||^2 + \rho_f ||g^+_f(x_f, x_{-f})||
\]

Set \( x^{k,f+1} = (x_1^{k+1}, \ldots, x_f^{k+1}, x_{f+1}^k, \ldots, x_F^k) \).
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IF

\[
\|\nabla_{x_f^{k+1}} \theta_f(x_f^{k+1}, x_{-f}) + 2\tau \|x_f^{k+1} - x_f^k\|_\gamma > c_f \left( \rho_f \left\| \nabla_{x_f^{k+1}} \|g^+_f(x_f^{k+1}, x_{-f})\|_\gamma \right\| \right)
\]

Update \( \rho_f \)
Convergence

- In [Facchinei et al. 11] (shared constraints GNEPs), for every $k$ and $f$, $x^{k,f}$ is feasible (to the GNEP) – we DON’T have it here.
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Strong (ordinal) potential function will guarantee that

$$\lim_{k \to \infty, k \in \mathcal{K}} x^{k,f} = \bar{x}, \forall f.$$
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- Exact penalization (finite penalty term upgrade): We say that the GNEP satisfies the EMFCQ at a point $\bar{x}$ if for every player $f = 1, \ldots, F$, there exists $d_{f}$ such that $\nabla_{x_{f}}g_{fi}(\bar{x}_{f}, \bar{x}_{-f})^{T}d_{f} < 0 \ \forall i \in I_{f}^{+}(\bar{x})$, where $I_{f}^{+}(\bar{x}) := \{i \in \{1, \ldots, m_{f}\} \mid g_{fi}(\bar{x}_{f}, \bar{x}_{-f}) \geq 0\}$ is the index set of all active and violated constraints at the point $\bar{x}$. 
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Workshop on Complementarity And Its Extensions, Singapore, 2012
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$$\nabla_{x_f} g_i(\bar{x}_f, \bar{x}_{-f})^T d_f < 0 \quad \forall i \in l_f^+(\bar{x}),$$

where

$$l_f^+(\bar{x}) := \{ i \in \{1, \ldots, m_f\} \mid g_i(\bar{x}_f, \bar{x}_{-f}) \geq 0 \}$$

is the index set of all active and violated constraints at the point $\bar{x}$.

**EMFCQ** holds at every cluster point $\bar{x} \implies$ Exact Penalization.
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Multiple Interdictors

Interdiction Games

We propose to study a class of decentralized network interdiction games that include multiple agents. In an interdiction problem, a defender seeks to destroy, neutralize, or delay its enemy's potential to launch effective attacks. Interdiction problems have been studied in a variety of military and homeland security contexts, such as coordinating tactical air strikes, combatting drug trafficking, and defending against the smuggling of nuclear material. For example, a supervising body may assign several agents to interdict different adversaries in a common system (such as unmanned aerial vehicles (UAVs) in a geographic network), in a system. However, in reality, planning is often performed from a centralized point of view, while execution is decentralized. There has been little work so far to study and address the loss in efficiency associated with decentralized decision-making in network interdiction situations, which will significantly advance the knowledge of decentralized decision-making in network interdiction.

Proposed research directions

Interdiction games are a class of games in which two or more players compete against each other in a network. Each player in an interdiction game has a set of actions that it can take to interdict the network, and each action has a cost associated with it. The goal of each player is to minimize its own cost while maximizing the cost to its opponent. Interdiction games are typically modeled as a game of imperfect information, where each player has incomplete information about the actions of its opponent. A decentralized algorithm for solving potential Generalized Nash Equilibrium Problems (GNEPs) numerically is proposed in this paper. The motivation and background of the research is presented in the introduction, followed by a motivating example and a discussion on the current state of the art in network interdiction games. The numerical studies section presents the results of applying the proposed algorithm to a variety of benchmark problems, and the conclusion and future work section summarizes the main findings of the research and outlines potential areas for future work.
Multiple Interdictors

Interdiction Games

- Multiple interdictors
Multiple Interdictors

Interdiction Games

- Multiple interdictors
- Each interdictor has its adversary (evader, follower, etc)
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Interdiction Games

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- Each interdictor has its adversary (evader, follower, etc)
- Each interdictor has its own objectives, interdicting costs and budgets
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- Multiple interdictors
- Each interdictor has its adversary (evader, follower, etc)
- Each interdictor has its own objectives, interdicting costs and budgets
- The interdictors share a network $G(N, A)$ (transportation network, communication network, supply chain network, etc)
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A Motivating Example

![Network Diagram]

Suppose there are two agents, each tasked with pro-

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loss. To fill this knowledge gap, we propose to establish theoretical foundations and computational frameworks 

missions in network-centric warfare environments have become increasingly common and important, we 

Interdiction problems have been studied in a variety of military and homeland security 

In an interdiction problem, a defender seeks to destroy, neutralize, or delay its enemy's potential to launch 

Interdiction Games

$
\begin{align*}
\text{Agent 1} \quad &\text{would increase the length of arc } (s_1, a) \text{ by 1 and Agent 2 would increase the length of arc } (s_2, a) \text{ by 1 unit,} \\
\text{Suppose the agents cannot} \quad &\text{coordinate with each other and must act independently (i.e., in a decentralized manner). In this case, Agent 1} \\
\text{incur the same interdiction costs, and both have an interdiction budget of} \quad &\text{1.1} \\
\text{Overall, this research will significantly advance the knowledge of decentralized decision-making in network interdiction situations,} \\
\text{and improve the Air Force's capability to rapidly and accurately make decisions and swiftly respond to} \\
\text{fast-changing environments faced by decentralized agents. It will also directly contribute to several priority} \\
\text{areas of the Air Force's Science and Technology Program, as listed in} \\
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\text{Interdicting parts of the system in order to achieve different, and potentially conflicting objectives. The} \\
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Shortest Path Interdiction Games

Bilevel game

Let \( d(a(x_1, \ldots, x_n)) \) denote the arc length after upper level interdiction. For each \( i = 1, \ldots, n \):

\[
\max_{x_i} f_i(x_i, x_i - x) \quad \text{s.t.} \quad \sum_{a \in A} c_i(a(x_i, x_i - x)) \leq B_i x_i \quad x_i \in X_i.
\]

where

\[
f_i(x_1, \ldots, x_n) = \begin{cases} \min \sum_{a \in p^*} d_a(x_1, \ldots, x_n) & \text{if } p^* \text{ is a } s_i-t_i \text{-path in aftermath network of } (x) \end{cases}
\]
Shortest Path Interdiction Games

Bilevel game

- **Upper level**: interdictor maximizing path length via $x$

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Shortest Path Interdiction Games

Bilevel game

- Upper level: interdictor maximizing path length via $x$
- Lower level: adversary finding the shortest path
Shortest Path Interdiction Games

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For each $i = 1, \ldots, n$:

$$\max_{x^i} f_i(x^i, x^{-i})$$

s.t. $\sum_{a \in A} c_a^i(x^i, x^{-i}) \leq B_i$

$x^i \in X^i$.

where

$$f_i(x^1, \ldots, x^n) = \begin{cases} \min_{a \in p^*} \sum_{a \in p^*} d_a(x^1, \ldots, x^n) \\ \text{s.t. } p^* \text{ is a } s_i-t_i \text{ path in aftermath network of } (x). \end{cases}$$
Shortest Path Interdiction Games – Reformulation

Using the dual of the lower-level shortest path problem, the bilevel program → one-level problem for each $i = 1, \ldots, n$.

\[
\begin{align*}
\text{max} \quad & y^i_t - y^i_s \\
\text{s.t.} \quad & y^i_v - y^i_u \leq d_{uv}(x^i, x^{-i}) \quad \text{for all } (u, v) \in A, \\
& \sum_{a \in A} c^i_a(x^i, x^{-i}) \leq B^i \\
& x^i \in X^i, \quad y^i_v \geq 0 \quad \text{for all } v \in V.
\end{align*}
\]
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\[
\begin{align*}
\max_{x^i, y^i} & \quad y^i_{t_i} - y^i_{s_i} \\
\text{s.t} & \quad y^i_v - y^i_u \leq d_{uv}(x^i, x^{-i}) \quad \text{for all } (u, v) \in A, \\
& \quad \sum_{a \in A} c^i_a(x^i, x^{-i}) \leq B_i \\
& \quad x^i \in X^i, \quad y^i_v \geq 0 \quad \text{for all } v \in V.
\end{align*}
\] (SPI)

- \( X^i = \mathbb{R}^A_{\geq 0}, \ i = 1, \ldots n, \) and \( d_a(x^i, x^{-i}) = d_a^0 + \sum_{i=1}^{n} x^i_a. \)
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- \( X^i = \mathbb{R}^A_{\geq 0}, \ i = 1, \ldots, n \), and \( d_a(x^i, x^{-i}) = d_a^o + \sum_{i=1}^n x^i_a \).
- \( X^i = \{0, 1\}^A, \ i = 1, \ldots, n \), and \( d_a(x^i, \ldots, x^{-i}) = d_a^o + e_a \max_{i \in N} x^i_a \).
An Example

\[ s_1 = \cdots = s_n = r_0 \]

![Figure 1: Network for example. Arc labels represent unit interdiction costs.](image)

- Figure 1: Network for example. Arc labels represent unit interdiction costs.
Another Example – Water Resource Management

\[
\text{maximize} \quad \theta_f(q_f, q_{-f}) = P(Q)q_f - c_f(q_f)
\]

subject to \[\sum_{h \in F} q_h \leq Q^{max} \].
Outline

1. Motivation and Background
2. A Decentralized Algorithm for Solving Potential GNEPs
3. Numerical Studies
4. Conclusion and Future Work
Conclusions

- Presented a decentralized algorithm to solve a class of (strong) potential GNEPs without shared constraints.
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- Could use decentralized algorithms to provide empirical or experimental justification for GNEP solutions.
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- Could use decentralized algorithms to provide empirical or experimental justification for GNEP solutions.

- The non-shared constraints cause theoretical difficulties for decentralized algorithms.
Future Work

- Other ways to deal with the non-shared constraints (e.g., Nash bargaining)
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- Large-population potential games
- Extend the classes of potential GNEPs
Thank you!

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References


