An effective form of Rademacher's theorem

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Plan of the talk

Rademacher's theorem

Lipschitz functions

A function f is called *Lipschitz* if for some constant K, for all x, y

$$\|f(x) - f(y)\| \leq K\|x - y\|$$

Theorem (Rademacher)

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, then it is almost everywhere differentiable.

- There are numerous extensions of this result: for different classes of functions (that exhibit some locally Lipschitz behaviour) and for different types of spaces.
- "Fréchet Differentiability of Lipschitz Functions and Porous Sets in Banach Spaces" by J. Lindenstrauss, D. Preiss and J. Tier, 2012.

The question of the converse

Theorem (Rademacher)

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function. Then there exists a null set, such that f is differentiable outside it.

The converse question

Let $N \subset \mathbb{R}^n$ be a null set. Is there a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ that is differentiable outside N?
(If the answer is no, characterise the family of null sets for which the answer is positive.)

- positive for $m = n = 1$;
- for $n \geq 2$ there is a null set N containing a point of differentiability of every Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ (Preiss 1980);
- A comprehensive answer has been given/announced very recently, not all relevant results have been published yet (Alberti, Csornyei, Preiss, Speight, Jones, etc);
- In general, the converse holds if and only if $m \geq n$.

Algorithmic randomness and differentiability

- Intuitively, an element is random if it does not exhibit any exceptional properties.
- This idea can be formalized by identifying exceptional properties with effective null sets.
- Different types of effective null sets correspond to different types of algorithmic randomness.
- Sufficiently well-behaved functions have small sets of non-differentiability $\rightarrow$ effective null-sets $\rightarrow$ tests for randomness.
- The notions relevant to this talk are: $WR, W2R, CR$

Rademacher's theorem in effective setting

A question that relates computable analysis and algorithmic randomness:

For a given class of effective functions to identify which randomness notion characterises the sets of differentiability.

- In the context of Rademacher's theorem, the goal is to characterize sets of non-differentiability of computable Lipschitz functions of several variables.

Define $CL_n = \{f : [0,1]^n \to \mathbb{R} : f \text{ is computable Lipschitz}\}$

- $RR_* = \{X \in 2^\omega : (\forall n)\ (\forall f \in CL_n)\ \ f'(0.X) \text{ exists}\}$
- $RR_n = \{x \in [0,1]^n : (\forall f \in CL_n)\ \ f'(x) \text{ exists}\}$
- previously known result: $RR_1 = CR$

Differentiability of Lipschitz functions on the unit interval

The answer is known, elegant and relatively easy:

- **Classically**: any null set is a set of non-differentiability of a Lipschitz function;
- **Effectively**: $CR \iff$ differentiability of computable Lipschitz functions.

Effective result was a bit surprising. A way to explain this:

- A Lipschitz function $\rightarrow$ a non-decreasing function $\rightarrow$ a computable measure $\rightarrow$ a computable martingale.

On the unit interval many results of this kind are known.

Differentiability of Lipschitz functions in higher dimensions

More difficult, in effective setting we have only a partial result.

- **Classically**: after more than 30 years of work, results constituting (what analysts consider) a comprehensive answer have been announced relatively recently. Not all have been published yet.
- **Effectively**: $CR \implies RR_n, RR_n \not\Rightarrow WR$,
  - $CR \iff RR_*$

There are relatively few results of this kind in higher dimensions.

Differentiability in higher dimensions

Fréchet derivative

We say f is *differentiable* at x if for some linear map T the following holds

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - T \cdot h}{||h||} = 0.$$

Then, by definition, $f'(x) = T$.

There are quite a few ways this condition can fail. This, and the fact that for different classes of functions these "fault lines" mean different things, are two of the reasons for exploring weaker notions of differentiability.

Weak Gatéaux $\implies$ **Gâteaux** $\implies$ Hadamard $\implies$ **Fréchet**

Gâteaux derivative

We define *one-sided directional derivative* as

$$D_+ f\,(x;v) = \lim_{t \downarrow 0} \frac{f(x+tv) - f(x)}{t}.$$

The *two-sided directional derivative* $Df\,(x;v)$ is defined by

$$Df\,(x;v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$

If all two-sided directional derivatives of f at x exist and $T = Df\,(x;\cdot)$ is linear, then f is said to be *Gâteaux-differentiable* at x and the linear map T is called the *Gâteaux derivative* of f at x.
$\overline{D}_+ f\,(x;v)$ and $\underline{D}_+ f\,(x;v)$ - upper and lower derivatives.
$D_i f(x)$ - partial derivative.

Differentiability of Lipschitz functions

There are two known facts that make our work with differentiability of Lipschitz functions much easier:

- For a Lipschitz function f, both $\overline{D}_+ f\,(x;\cdot)$ and $\underline{D}_+ f\,(x;\cdot)$ are continuous everywhere.
- Gâteaux differentibility $\implies$ Fréchet differentiability

The important for us consequences of the above facts are following:

- when working with directional derivatives, it is sufficient to consider computable directions only, and
- to show differentiability of f at x, it is sufficient to show that $v \mapsto D_+ f\,(x;v)$ is defined and is linear (on computable v).

Properties of computable randomness (1)

$z = (0.Z_1, \ldots, 0.Z_n) \in [0,1]^n$ is computably random if $Z_1 \oplus \cdots \oplus Z_n$ is computably random.

We will rely on two preservation properties of CR.

Definition

A total computable function $d : 2^\omega \times 2^{<\omega} \to \mathbb{R}$ is a *uniform computable martingale* if $d(Z, \cdot)$ is a martingale for every $Z \in 2^\omega$.
We say A is *computably random uniformly relative to* B if there is no uniform computable martingale d such that $d(B, \cdot)$ succeeds on A.

Note that the above definition works for elements of $[0,1]^n$ as well.

Theorem (Rute,Miyabe)

$A \oplus B$ is computably random if and only if A is computably random uniformly relative to B and B is computably random uniformly relative to A.

Properties of computable randomness (2)

We say that $\phi : [0,1]^n \to [0,1]^n$ is *a.e. computable*, if there exists a partial computable $F : \mathbb{N}^\omega \to \mathbb{N}^\omega$ and a subset $A \subseteq [0,1]^n$ of full measure, such that:

1. for all $x \in A$, given a Cauchy name of x, F computes a Cauchy name for $\phi(x)$, and
2. $x \in A$ iff for all a, b, which are Cauchy names for x, both $F(a)$ and $F(b)$ are Cauchy names for the same element.

We say that $\phi : [0,1]^n \to [0,1]^n$ is an *a.e. computable isomorphism*, if there exists $\psi : [0,1]^n \to [0,1]^n$ such that $\phi \circ \psi = \text{id}$ and $\psi \circ \phi = \text{id}$ almost everywhere and both ψ, ϕ are measure preserving and a.e. computable.

Theorem (Rute)

Computable randomness is preserved by a.e. computable isomorphisms.

$CR \implies RR_n$: the proof

Theorem

Let $z \in [0,1]^n$ be computably random and let $f : [0,1]^n \to \mathbb{R}$ be a computable Lipschitz function. Then f is differentiable at z.

Let $f : [0,1]^n \to \mathbb{R}$ be a computable $K-$Lipschitz function and let z be CR. We will show that $f'(z)$ exists. The proof has three steps:

- all partial derivatives exist
- all directional derivatives for computable directions exist
- $Df(z;\cdot)$ is linear on computable directions

This is enough to show that f Gâteaux-differentiable at z and since f is Lipschitz, we're done.

Step 1: existence of partial derivatives

First, note that $x \mapsto f(x) + \boldsymbol{K} \cdot x$, where $\boldsymbol{K} = (K, \ldots, K) \in \mathbb{R}^n$, is a computable non-decreasing function and its partial derivatives exist whenever partial derivatives of f exist. Hence in this step we may assume f is non-decreasing.

- Fix $i \leq n$. Let $y = z - z_i e_i$ and let Y be its binary expansion.
- Consider $g(h) = f(y + he_i)$. Note that $g'(z_i) = D_i f(z)$. By contraposition, suppose $g'(z_i)$ does not exist. We need to show that Z_i is not computably random uniformly relative to Y.
- We provide the required uniform computable martingale by adapting the proof for 1-dimensional case by Nies.

Step 2: existence of directional derivatives

Two simple facts to consider:

- computable randomness is invariant under computable linear isometries, and
- linear functions are Lipschitz.

Fix a computable v and let $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ denote a change of basis map such that $\Theta(e_1) = v$. We may assume it is computable.
Consider $g = f \circ \Theta$. It is a computable Lipschitz function.
Moreover, $D_+f\,(z\,;v) = D_+g\,(z'\,;e_1) = D_1g(z')$, where is a computable linear image of z. We're done, since z' is CR, and g is a computable Lipschitz function.

Step 3: linearity of $Df\,(z;\cdot)$

For $x \in [0,1]^n$, $v \in \mathbb{R}^n$, and $h \in \mathbb{R}$, define $\delta_f^v(x,h) = \frac{f(x+hv)-f(x)}{h}$.
Fix some computable v, u and suppose

$$|Df\,(z;u) + Df\,(z;v) - Df\,(z;u+v)| > q$$

for some rational q. Then there is $p > 0$ such that

$$\left|\delta_f^v(z,h) + \delta_f^u(z,h) - \delta_f^{v+u}(z,h)\right| \geq q$$

for all $h \leq p$. Hence z belongs to the following set:

$$\left\{x : \forall h\ \left(h \leq p \implies \left|\delta_f^v(x,h) + \delta_f^u(x,h) - \delta_f^{v+u}(x,h)\right| \geq q\right)\right\},$$

which is a Π_1^0 null set.

$\mathrm{RR}_n \not\Rightarrow$ WR

Theorem (Doré, Maleva (2011))

There is a compact null-set $M(n)$ that contains points of differentiability of all Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$.

Construction by Doré and Maleva.

- $(N_i)_{i \in \mathbb{N}}$ - odd integers with $N_1 > 1$, $\lim_i N_i = \infty$ and $\sum \frac{1}{N_i^2} = \infty$.
- $(p_i)_{i \in \mathbb{N}}$ - reals with $1 \leq p_i \leq N_i$ and $\lim_i p_i / N_i = 0$.
- Let $d_0 = 1$ and for all $i \geq 1$ let $d_i = \prod_{k \leq i} N_k^{-1}$ and define a lattice in $\mathbb{R}^2$

$$C_i = \left(\frac{d_{i-1}}{2}, \frac{d_{i-1}}{2} \right) + \mathbb{Z}^2.$$

Finally, define

$$W = \mathbb{R}^2 \setminus \bigcup_{i \geq 1} \bigcup_{c \in C_i} \mathcal{B}_\infty(c, p_i d_i / 2),$$

where $\mathcal{B}_\infty(x, r)$ denotes an open ball in $(\mathbb{R}^2, \| \cdot \|_\infty)$.

$\mathrm{RR}_n \not\Rightarrow \mathrm{WR}$

- W is a closed null set. It is possible to take all parameters computable and make it Π^0_1.
- Doré and Maleva proved [Corollary 5.2] that for any such W, any open neighbourhood of the set $M(n) = \mathbb{R}^{n-2} \times W$ contains a point of differentiability of any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$.
- In particular, $[0,1]^n \cap M(n)$ contains a point of differentiability of any Lipschitz $f : [0,1]^n \to \mathbb{R}$.
- However, if $\mathcal{C} \subseteq [0,1]^n$ is porous and Π^0_1, then $x \mapsto d(\mathcal{C}, x)$ is a computable Lipschitz functions not differentiable at $\mathcal{C}$.
  $RR_n \implies WR^p$

We do know:

- $RR_* = RR_1 = CR$
- $CR \implies RR_n \implies WR^p$, $RR_n \not\Longrightarrow WR$

Open question

To find a satisfactory characterisation of RR_n for $n > 1$.