A computable absolutely normal Liouville number

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The set of *Liouville numbers* is $\{x \in \mathbb{R}\backslash\mathbb{Q} : \forall k \in \mathbb{N},\ \exists q \in \mathbb{N},\ q > 1 \text{ and } ||qx|| < q^{-k}\}$ where $||x|| = \min\{|x-m| : m \in \mathbb{Z}\}$ is the distance of a real number x to the nearest integer and other notation is as usual. Liouville's constant, $\sum_{k\geqslant 1} 10^{-k!}$, is the standard example of a Liouville number. Though uncountable, the set of Liouville numbers is small, in fact, it is null, both in Lebesgue measure and in Hausdorff dimension (see [2]).

We say that a *base* is an integer s greater than or equal to 2. A real number x is *normal to base* s if the sequence $(s^j x : j \geqslant 0)$ is uniformly distributed in the unit interval modulo one. *Absolute normality* is normality to every base. Bugeaud [2] established the existence of absolutely normal Liouville numbers, but does not provide a construction. Let's recall that a real number x is *computable* if there is a base s and an algorithm to output the digits for the base-s expansion of x, one after the other. We show the following:

Theorem. *There is a computable absolutely normal Liouville number.*

We regard this result as a step into the ancient problem posed by Émile Borel [1] on exhibiting a natural instance of an absolutely normal number. Borel's understanding of "natural" may have been towards numbers that can be described geometrically (as π), analytically (as e), or algebraically (as $\sqrt{2}$). To our mind, algorithmic descriptions are also explicit, immediate and worthy of investigation.

[1] Émile Borel. Les probabilités dénombrables et leurs applications arithmétiques. *Supplemento di Rendiconti del circolo matematico di Palermo*, 27:247–271, 1909.

[2] Yann Bugeaud. Nombres de Liouville et nombres normaux. *Comptes Rendus de l'Académie des Sciences Paris*, 335(2):117–120, 2002.