An introduction to provability degrees

Mingzhong Cai

June 2014
CCR Singapore

(partially joint with Andrews, Diamondstone, Lempp and Miller)

In the literature, there are classical independence results of Π^0_2 (Π^0_1) sentences over Peano Arithmetic:

- Gödel's Incompleteness Theorem (consistency of PA)
- Paris-Harrington Theorem (a modified version of finite Ramsey Theorem)
- Paris-Kirby Theorem (Goodstein's Theorem)

Remark

The following are equivalent:

- Φ is a Π^0_2 sentence (in arithmetic);
- Φ is a statement that some partial recursive function f is total (denoted as $tot(f)$).

Remark

The following are equivalent:

- Φ is a Π^0_2 sentence (in arithmetic);
- Φ is a statement that some partial recursive function f is total (denoted as $tot(f)$).

Remark: Base theory

Our base theory, denoted as T, is (recursively) axiomatizable, interprets arithmetic and is sound. For convenience we also require that T extends $\mathbf{I\Sigma}_1$. Examples of such base theories: **PA**, **ZFC**.

Definition

Given two total recursive functions (with fixed algorithms) φ and ψ, we say φ is provably reducible to ψ, if $T + \mathit{tot}(\psi) \vdash \mathit{tot}(\varphi)$.

Definition

Given two total recursive functions (with fixed algorithms) φ and ψ, we say φ is provably reducible to ψ, if $T + tot(\psi) \vdash tot(\varphi)$.

Convention

In studying the totality of partial recursive functions, we assume that their domains are initial segments of ω (using $\mathbf{I\Sigma}_1$).

Definition

Given two total recursive functions (with fixed algorithms) φ and ψ, we say φ is provably reducible to ψ, if $T + tot(\psi) \vdash tot(\varphi)$.

Convention

In studying the totality of partial recursive functions, we assume that their domains are initial segments of ω (using $\mathbf{I\Sigma}_1$).

Remark

This preorder induces a degree structure (provability degrees) on all total recursive algorithms. We use $[\varphi]$ to denote the degree of φ. Essentially, this is Lindenbaum algebra on all true Π^0_2 sentences (in arithmetic).

Why do we study this structure?

- It provides a framework easy for recursion theorists to work on proof-theoretic problems, particularly without too much restraint on the base theory.
- A number of recursion-theoretic concepts and construction ideas can be used to study this proof-theoretic structure.
- It helps to clarify the difference between truth and provability.

Remark: [0]

We use [0] to denote the bottom degree, i.e., the class of T-provably total algorithms. (Here 0 is the constant zero function, computed in the easiest way.)

Remark: [0]

We use [0] to denote the bottom degree, i.e., the class of T-provably total algorithms. (Here 0 is the constant zero function, computed in the easiest way.)

Density Theorem, unrelativized [ACDLM]

For every $[\varphi] > [0]$, there is a $[\psi] \in ([0], [\varphi])$.

Idea from recursion theory

Split one big requirement into small subrequirements and try to satisfy them step by step in a construction. Example: in Kleene-Post theorem, one requirement $A \not\leq_T B$ is split into subrequirements $\Phi_e(B) \neq A$ and these subrequirements are satisfied step by step in the construction.

Idea from recursion theory

Split one big requirement into small subrequirements and try to satisfy them step by step in a construction. Example: in Kleene-Post theorem, one requirement $A \not\leq_T B$ is split into subrequirements $\Phi_e(B) \neq A$ and these subrequirements are satisfied step by step in the construction.

List of Requirements

- For each s, a code of a proof, s does not witness the fact that $T \vdash tot(\psi)$;
- For each s, a code of a proof, s does not witness the fact that $T + tot(\psi) \vdash tot(\varphi)$.

Proof of Density Theorem

We construct (compute) a recursive function ψ (using Recursion Theorem):

- At stage $2s$, we check if s is a proof witnessing $T \vdash tot(\psi)$. If not, let $\psi(2s) = 0$;

Proof of Density Theorem

We construct (compute) a recursive function ψ (using Recursion Theorem):

- At stage $2s$, we check if s is a proof witnessing $T \vdash tot(\psi)$. If not, let $\psi(2s) = 0$; if so, let $\psi(t) = \varphi(t)$ for all $t \geq 2s$.

Proof of Density Theorem

We construct (compute) a recursive function ψ (using Recursion Theorem):

- At stage $2s$, we check if s is a proof witnessing $T \vdash tot(\psi)$. If not, let $\psi(2s) = 0$; if so, let $\psi(t) = \varphi(t)$ for all $t \geq 2s$.
- At stage $2s + 1$, we check if s is a proof witnessing $T + tot(\psi) \vdash tot(\varphi)$. If not, let $\psi(2s + 1) = 0$; if so, let $\psi(t) = 0$ for all $t \geq 2s + 1$.

Proof of Density Theorem

We construct (compute) a recursive function ψ (using Recursion Theorem):

- At stage $2s$, we check if s is a proof witnessing $T \vdash tot(\psi)$. If not, let $\psi(2s) = 0$; if so, let $\psi(t) = \varphi(t)$ for all $t \geq 2s$.
- At stage $2s + 1$, we check if s is a proof witnessing $T + tot(\psi) \vdash tot(\varphi)$. If not, let $\psi(2s + 1) = 0$; if so, let $\psi(t) = 0$ for all $t \geq 2s + 1$.

Verification

We show that "if so" never happens in the above construction:

- If at some stage $2s$, "if so" triggers us to make $\psi(t) = \varphi(t)$, then T proves $tot(\psi)$ which is now equivalent to $tot(\varphi)$. We get a contradiction.
- If at some stage $2s + 1$, "if so" triggers us to make $\psi(t) = 0$, then T now proves $tot(\psi)$ and consequently $tot(\varphi)$. Again we get a contradiction.

Operations on the degrees

- Join and meet: conjunction and disjunction
- two jump-like operators: hop and jump

Operations on the degrees

- Join and meet: conjunction and disjunction
- two jump-like operators: hop and jump

Hop and Skip (Consistency)

Given a degree $[\varphi]$, by incompleteness theorem, $T + tot(\varphi) \nvdash Con(T + tot(\varphi))$. So $[\varphi] \vee [Con(T + tot(\varphi))]$, the hop of $[\varphi]$ (denoted as $[\varphi]^\circ$), is a degree strictly higher than $[\varphi]$. $[Con(T + tot(\varphi)]$, or $[con(\varphi)]$, is called the skip of $[\varphi]$

Jump (Σ_1-Soundness)

Given a function φ, we define φ^*, the jump of φ as follows:

$$\varphi^*(s) = \begin{cases} \varphi_e(s) + 1, & \text{if } s : T + tot(\varphi) \vdash tot(\varphi_e); \\ 0, & \text{otherwise.} \end{cases}$$

One can show that $tot(\varphi^*)$ is the equivalent to the Σ_1-soundness (1-consistency) of $T + tot(\varphi)$, and it is automatic that $[\varphi] < [\varphi^*]$. In addition, this operator is degree invariant, and so induces a jump in the degrees.

Jump (Σ_1-Soundness)

Given a function φ, we define φ^*, the jump of φ as follows:

$$\varphi^*(s) = \begin{cases} \varphi_e(s) + 1, & \text{if } s : T + tot(\varphi) \vdash tot(\varphi_e); \\ 0, & \text{otherwise.} \end{cases}$$

One can show that $tot(\varphi^*)$ is the equivalent to the Σ_1-soundness (1-consistency) of $T + tot(\varphi)$, and it is automatic that $[\varphi] < [\varphi^*]$. In addition, this operator is degree invariant, and so induces a jump in the degrees.

Proposition

$[\varphi] < [\varphi]^\circ < [\varphi]^*$.

Jump (Σ_1-Soundness)

Given a function φ, we define φ^*, the jump of φ as follows:

$$\varphi^*(s) = \begin{cases} \varphi_e(s)+1, & \text{if } s : T + tot(\varphi) \vdash tot(\varphi_e); \\ 0, & \text{otherwise.} \end{cases}$$

One can show that $tot(\varphi^*)$ is the equivalent to the Σ_1-soundness (1-consistency) of $T + tot(\varphi)$, and it is automatic that $[\varphi] < [\varphi^*]$. In addition, this operator is degree invariant, and so induces a jump in the degrees.

Proposition

$[\varphi] < [\varphi]^\circ < [\varphi]^*$.

Hop/Jump Inversion [ACDLM]

For every $[\varphi] \geq [0]^\circ([0]^*)$, there is a degree $[\theta]$ such that $[\varphi] = [\theta]^\circ([\theta]^*)$.

difference between the hop and the jump:

difference between the hop and the jump:

Proposition

$[0]^\circ$ is a half of a minimal pair.

difference between the hop and the jump:

Proposition

$[0]^{\circ}$ is a half of a minimal pair.

Theorem [ACDLM]

$[0]^{*}$ is not a half of a minimal pair.

Definition

A degree is Π^0_1 if it is equivalent to a Π^0_1 sentence over the base theory.

Definition

A degree is Π^0_1 if it is equivalent to a Π^0_1 sentence over the base theory.

Skip Inversion [ACDLM]

For every Π^0_1 degree $[\varphi] \geq [0]^\circ$, there is a Π^0_1 degree $[\theta] < [\varphi]$ such that $[\varphi] = [con(\theta)]$.

Definition

Given a nonzero degree $[\varphi]$, the unprovability of $[\varphi]$, denoted as $[\varphi]^{-1}$, is the degree of the Π^0_1 sentence "$T \nvdash tot(\varphi)$".

Definition

Given a nonzero degree $[\varphi]$, the unprovability of $[\varphi]$, denoted as $[\varphi]^{-1}$, is the degree of the Π^0_1 sentence "$T \nvdash tot(\varphi)$".

Unprovability of $[0]^\circ = [con(0)]$ or higher degrees

$[con(0)]^{-1} = [con(0)]$, and for every $[\varphi] \geq [con(0)]$, $[\varphi]^{-1} = [con(0)]$.

Definition

Given a nonzero degree $[\varphi]$, the unprovability of $[\varphi]$, denoted as $[\varphi]^{-1}$, is the degree of the Π^0_1 sentence "$T \nvdash tot(\varphi)$".

Unprovability of $[0]^\circ = [con(0)]$ or higher degrees

$[con(0)]^{-1} = [con(0)]$, and for every $[\varphi] \geq [con(0)]$, $[\varphi]^{-1} = [con(0)]$.

Theorem

$[\theta]$ is a Π^0_1 degree above $[0]^\circ$ if and only if $[\theta] = [\varphi]^{-1}$ for some $[\varphi]$.

Hilbert's Program:

Find an axiom system T such that for every true arithmetic sentence P, T proves P.

Hilbert's Program:

Find an axiom system T such that for every true arithmetic sentence P, T proves P.

Gödel:

This is a dream.

Hilbert's Program, Modified:

Find an axiom system T such that for every true arithmetic sentence P,

- T proves P,

Hilbert's Program, Modified:

Find an axiom system T such that for every true arithmetic sentence P,

- T proves P, or
- prove $T \nvdash P$

Hilbert's Program, Modified:

Find an axiom system T such that for every true arithmetic sentence P,

- T proves P, or
- $T + con(T)$ proves $T \nvdash P$.

Hilbert's Program, Modified:

Find an axiom system T such that for every true arithmetic sentence P,

- T proves P, or
- $T + con(T)$ proves $T \nvdash P$.

Remark

Yet another dream.

The Ultimate Hilbert's Program:

Find an axiom system T such that for every true arithmetic sentence P,

- T proves P, or
- $T + con(T)$ proves $T \nvdash P$, or
- $T + con(con(T))$ proves $T + con(T) \nvdash$ "$T \nvdash P$", or

$$\vdots$$

The Ultimate Hilbert's Program:

Find an axiom system T such that for every true arithmetic sentence P,

- T proves P, or
- $T + con(T)$ proves $T \nvdash P$, or
- $T + con(con(T))$ proves $T + con(T) \nvdash$ "$T \nvdash P$", or

$$\vdots$$

Question

A Hilbert's nightmare is a true sentence P for which the above list fails. Does such sentence exist?

The Ultimate Hilbert's Program, Phase II:

Find an uniform list of axiom systems T_i ($i \in \omega$), where each T_{i+1} extends $T_i \cup con(T_i)$ and such that for every true arithmetic sentence P,

- T_0 proves P, or
- T_1 proves $T_0 \nvdash P$, or
- T_2 proves $T_1 \nvdash$ "$T_0 \nvdash P$", or

$$\vdots$$

The Ultimate Hilbert's Program, Phase II:

Find an uniform list of axiom systems T_i $(i \in \omega)$, where each T_{i+1} extends $T_i \cup con(T_i)$ and such that for every true arithmetic sentence P,

- T_0 proves P, or
- T_1 proves $T_0 \nvdash P$, or
- T_2 proves $T_1 \nvdash$ "$T_0 \nvdash P$", or

$$\vdots$$

Remark

If we drop the requirement of being uniform, then such list exists.

Thank you!