Computations with Incomplete or Imperfect Information

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In 2012, Jockusch, and Schupp introduce and analyze the notion of generic computability. Informally, real is generically computable if there is a computation of that real that is usually correct.

We formalize our notion of “usually” using asymptotic density:

**Definition**

The density of real $A$ is the limit of the densities of its initial segments, $\lim_{n \to \infty} \frac{|A \cap \mathbb{N}|}{n}$. 
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**Computations with Incomplete or Imperfect Information**
Generic and coarse computability

**Definition**

A real $A$ is **generically computable** if there exists a partial computable function $\varphi$ whose domain has density 1 such that $\varphi(n) = A(n)$ for all $n \in \text{dom}(\varphi)$.

This is distinct from the following related notion.

**Definition**

A real $A$ is **coarsely computable** if there exists a total computable function $\varphi$ such that $\{n : \varphi(n) = B(n)\}$ has density 1.

So a generic computation is a computation that usually halts, always correctly, while a coarse computation is a computation that always halts, usually correctly.
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**Computations with Incomplete or Imperfect Information**
Theorem (Jockusch, Schupp, 2012)

*Neither generic computability, nor coarse computability implies the other.*

Objection (Moral Grounds)

*It is better to be incomplete than to be inaccurate!*

Metatheorem

*Generic computability is closer to coarse computability than coarse computability is to generic computability.*
Examples

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How does one produce a coarsely computable real that is not generically computable?

Any real that has density 1 is coarsely computable – just make sure each potential generic computation is wrong at least once.

Theorem (I, 2013)

Every nonzero Turing degree computes a real that is coarsely computable but not generically computable.
Coarse but not Generic

How does one produce a coarsely computable real that is not generically computable?

Any real that has density 1 is coarsely computable – just make sure each potential generic computation is wrong at least once.

**Theorem (I, 2013)**

*Every nonzero Turing degree (Turing) computes a real that is coarsely computable but not generically computable.*
So, how does one produce something generically computable but not coarsely computable? (Proof sketch)

**Theorem (Downey, Jockusch, Schupp, 2013)**

*Every nonzero c.e. degree (Turing) computes a real that is generically computable but not coarsely computable.*

**Theorem (Hirschfeld, Jockusch, McNicholl, Schupp)**

*If A is 1-generic, or weakly 2-random, then A does not compute any sets that are generically computable but not coarsely computable.*
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This addresses the question of how difficult it is to witness a nonimplication, but now we ask how far the nonimplications can be pushed.

Let \( r \in [0, 1] \).

**Definition**

A real \( A \) is generically computable at density \( r \) if there exists a partial computable function \( \varphi \) whose domain has lower density \( \geq \alpha \) such that \( \varphi(n) = A(n) \) for all \( n \in \text{dom}(\varphi) \).

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Theorem (Downey, Jockusch, McNicholl, Schupp)

If $A$ is generically computable at density $r$, then for every $\epsilon > 0$, $A$ is coarsely computable at density $r - \frac{\epsilon}{2}$.

Proof: Nonuniformly give yourself the point at which the density of the domain of the generic computation never again drops below $r - \frac{\epsilon}{2}$.

Observation

There exist reals that are coarsely computable, but not generically computable at any positive density. (i.e. coarsely computable, and absolutely undecidable.)
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For a recursion theorist, probably the most natural way of asking how close something is to being computable is by asking about its Turing degree.

If we wish to know how close a real is to being generically or coarsely computable, we should ask the question within a degree structure for that computability.

We now introduce generic reducibility, and coarse reducibility.
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We now introduce generic reducibility, and coarse reducibility.
Partial Oracles

**Definition**

Let $A$ be a real. Then a (time-dependent) **partial oracle**, $(A)$, for $A$ is a set of ordered triples $\langle n, x, s \rangle$ such that:

$\exists s (\langle n, 0, s \rangle \in (A)) \implies n \not\in A,$

$\exists s (\langle n, 1, s \rangle \in (A)) \implies n \in A.$

We think of $(A)$ as a partial function, sending $n$ to $x$. We think of $s$ as the number of steps it takes $(A)$ to converge.

The **domain** of $(A)$ is the set of $n$ for which there exists such an $x, s$. 
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Definition

Let $A$ be a real. Then a generic oracle for $A$ is a partial oracle whose domain is density-1.

Note that generically computing $A$ is equivalent to computing a generic oracle for $A$.

Definition

Let $A, B$ be reals. We say $A$ is generically reducible to $B$ (or $A \leq_g B$) if there is a Turing functional $\varphi$ such that for every generic oracle $(B)$, for $B$, $\varphi^{(B)}$ is a generic computation of $A$. 
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Coarse reductions

Definition
Let $A$ be a real. Then a coarse oracle for $A$ is an (ordinary Turing) oracle for a set that agrees with $A$ on density-1.

Definition
Let $A, B$ be reals. We say $A$ is coarsely reducible to $B$ (or $A \leq_g B$) if there is a Turing functional $\varphi$ such that for every coarse oracle $(B)$, for $B$, $\varphi^{(B)}$ is a coarse computation of $A$. 

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Computations with Incomplete or Imperfect Information
There is a natural embedding of the Turing degrees into the generic degrees:

**Definition**

For any real $X$, let $\mathcal{R}(X)$ be defined as follows.

$$\mathcal{R}(X) = \{2^n(2k + 1) : n \in X\}.$$

So we have “stretched” every bit of $X$ into a positive density “column” of $\mathcal{R}(X)$.

Since every generic computation of $\mathcal{R}(X)$ must include at least one bit from every column, it must be able to compute $X$.

As a result, generically computing $\mathcal{R}(X)$ is the same as computing $X$, and working with $\mathcal{R}(X)$ as a generic oracle is the same as working with $X$ as an oracle in the usual sense.
Embedding the Turing degrees in the generic degrees

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Note that this embedding fails quite badly for the coarse degrees.

**Observation**

*If $A$ is $\Delta^0_2$, then $R(A)$ is coarsely computable.*

**Theorem (Hirschfeldt, Jockusch, Kuyper, Schupp), (Dzhafarov, I.)**

*There exists an embedding of the Turing degrees into the coarse degrees.*
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**Theorem (Hirschfeldt, Jockusch, Kuyper, Schupp), (Dzhafarov, I.)**

*There exists an embedding of the Turing degrees into the coarse degrees.*
We say that a generic degree is density-1 if it is the generic degree of a density-1 real.

Lemma

The density-1 generic degrees are precisely the generic degrees of the coarsely computable reals.

Proof: Consider the set of $n$ on which the coarse computation is correct.
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Lemma

The density-1 generic degrees are precisely the generic degrees of the coarsely computable reals.

Proof: Consider the set of $n$ on which the coarse computation is correct.
**Theorem (I.)**

Let $A$ be a real. Then $A$ is hyperarithmetic if and only if there is a density-1 real $B$, such that $B \geq_{g} \mathcal{R}(A)$.

This uses Solovay’s characterization of the hyperarithmetic reals in terms of moduli of computation.

**Theorem (Solovay)**

Let $A$ be a real. Then $A$ is hyperarithmetic if and only if there is a function $f$, and a Turing functional $\varphi$ such that for every function $g$ majorizing $f$, $\varphi^{g}$ is a computation of $A$. In this case, we say that $f$ is a modulus of computation for $A$. 
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Idea: Let $B$ be a density-1 real. Then the rate at which the density of $B$ goes to $1$ is a slow growing function, and any generic oracle for $B$ computes a slower growing function.

This gives us one direction immediately: any modulus of computation can be emulated by the generic degree of a coarsely computable real.

**Objection**

Just because you know how quickly the density goes to $1$ doesn’t mean you know exactly which elements are missing!
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Theorem (I.)

Let $A$ be a real. Then $A$ is hyperarithmetic if and only if there is a density-1 real $B$, such that $B \geq g \mathcal{R}(A)$.

($\Rightarrow$) This direction is easy:
Make the density of $B$ approach 1 very slowly. Then any generic oracle will have density that also approaches 1 at least as slowly.
Theorem (I.)

Let $A$ be a real. Then $A$ is hyperarithmetic if and only if there is a density-1 real $B$, such that $B \geq_R \mathcal{R}(A)$.

$(\Rightarrow)$ This direction is easy:
Make the density of $B$ approach 1 very slowly. Then any generic oracle will have density that also approaches 1 at least as slowly.
That’s totally correct!

**Theorem (I.)**

There exists a density-1 real, $B$, such that for every $f : \mathbb{N} \to \mathbb{N}$, and every $\varphi$, there is a $g \geq f$ such that $\varphi^g$ is not a generic computation of $B$.

However, the rate of growth of $B$ can be used to compute any Turing degree that embeds below $B$. 
Start with $B \geq_g \mathcal{R}(A)$
- Choose $f$ so that for any $g \gg f$, $g$ can generate a tree of density-1 oracles that includes $B$.
- Those oracles then repeatedly attempt to elect a “leader” who can cause them to vote unanimously.
- $B$ is such a leader, so eventually they will find one.
- $B$ always votes correctly, so when they find a leader, the vote will be correct.

Note that intersecting $B$ with a density-1 real provides a generic oracle for $B$. 
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Question

What about the coarse degrees of generically computable reals? Is it possible to code any Turing information into such a degree?
We ask one last question

**Question**

*Given a nonzero generic degree $a$, is there always a density-1 degree $b$ such that $a \geq_g b$?*

If the answer to the question is “yes,” then there cannot be any minimal generic degrees, because the density-1 degrees are dense.

If the answer to the question is “no,” then the counterexample is half of a minimal pair for generic reduction.
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If the answer to the question is “no,” then the counterexample is half of a minimal pair for generic reduction.
Thank you for your attention.

