Weak Lowness Notions and Weak Reducibilities

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June 13, 2014

Weak Lowness Notions

Question

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- It should be closed downward under $\leq_T$.

Definition (Chaitin; Solovay)

A real A is *K-trivial* if for all n, $K(A\restriction_n) \leq^+ K(n)$.

Definition (Muchnik)

A real A is *low for K* if for all σ, $K(\sigma) \leq^+ K^A(\sigma)$.

- K-trivial $\Leftrightarrow$ Low for K $\Leftrightarrow$ Low for MLR (Nies 2005).
- The K-trivials are closed downward under $\leq_T$ (Hirschfeldt, Nies 2005).
- The K-trivials are closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003).
- There are only countably many K-trivials, and they are all Δ^0_2 (Chaitin, 1976).
- The K-trivials are all low, and in fact superlow (Nies 2005).

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One Way to Weaken

Definition (Barmpalias, Vlek)

A real A is *infinitely often K-trivial* if for infinitely many n, $K(A\upharpoonright_n) \leq^+ K(n)$.

Definition (Miller)

A real A is *weakly low for K* if for infinitely many σ, $K(\sigma) \leq^+ K^A(\sigma)$.

Infinitely Often K-Triviality

Theorem (Barmpalias, Vlek)

- *Every r.e. set is i.o. K-trivial.*
- *Every $\leq_{tt}$-degree contains an i.o. K-trivial.*
- *There is a perfect set of i.o. K-trivials.*
- *Every set that is computed by a 1-generic is i.o. K-trivial.*
- *No Martin-Löf random set is i.o. K-trivial.*

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Weak Lowness for K

Theorem (Miller)

A is weakly low for K iff A is low for Ω, i.e. $\Omega = \mu(dom(\mathbb{U}))$ is ML-random relative to A.

Corollary (via Nies, Stephan, Terwijn)

A is 2-random iff A is ML-random and weakly low for K.

Proposition

- *Weakly Low for K is closed downward under $\leq_T$.*
- *If A is weakly low for K then it is GL_1 ($A' \equiv_T A \oplus \emptyset'$) (Nies, Stephan, Terwijn).*
- *If A is Δ^0_2 and weakly low for K, then A is low for K (follows from Hirschfeldt, Nies, Stephan).*

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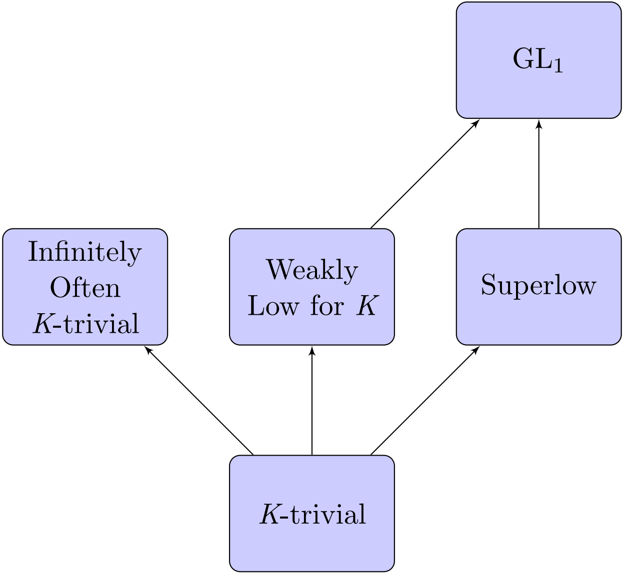
Picture So Far



Another Way to Weaken

Definition

A is Δ^0_2-*bounded* K-*trivial* if for all n, $K(A\restriction_n) \leq^+ K(n) + f(n)$ for all Δ^0_2 orders f.

Definition

A is Δ^0_2-*bounded low for* K if for all σ, $K(\sigma) \leq^+ K^A(\sigma) + f(\sigma)$ for all Δ^0_2 orders f.

We use $\mathcal{KT}(\Delta^0_2)$ and $\mathcal{LK}(\Delta^0_2)$ to denote these sets of reals.

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We use $\mathcal{KT}(\Delta^0_2)$ and $\mathcal{LK}(\Delta^0_2)$ to denote these sets of reals.

Δ_2^0-Bounded Notions

- $\mathcal{LK}(\Delta_2^0) \Rightarrow \mathcal{KT}(\Delta_2^0)$, but the implication does not reverse (H. 2013).
- $\mathcal{LK}(\Delta_2^0)$ contains a perfect set. (H. 2013)
- $\mathcal{LK}(\Delta_2^0)$ is closed downward under $\leq_T$, but for any real A, there is a $B \in \mathcal{KT}(\Delta_2^0)$ with $A \leq_T B$. (H. 2013)
- $\mathcal{KT}(\Delta_2^0)$ is closed under effective join, but for any real A, there are $B, C \in \mathcal{LK}(\Delta_2^0)$ with $A \leq_T B \oplus C$. (H.)
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Comparisons

- Δ^0_2-bounded K-trivial implies Infinitely Often K-trivial, but the implication does not reverse.
- Neither of Δ^0_2-bounded Low for K and Weakly Low for K implies the other.
- If A is Δ^0_2, then $A \in \mathcal{KT}(\Delta^0_2) \Leftrightarrow A \in \mathcal{LK}(\Delta^0_2) \Leftrightarrow A$ is K-trivial.
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ Low for Effective Dimension. (Hirshfeldt, Weber)
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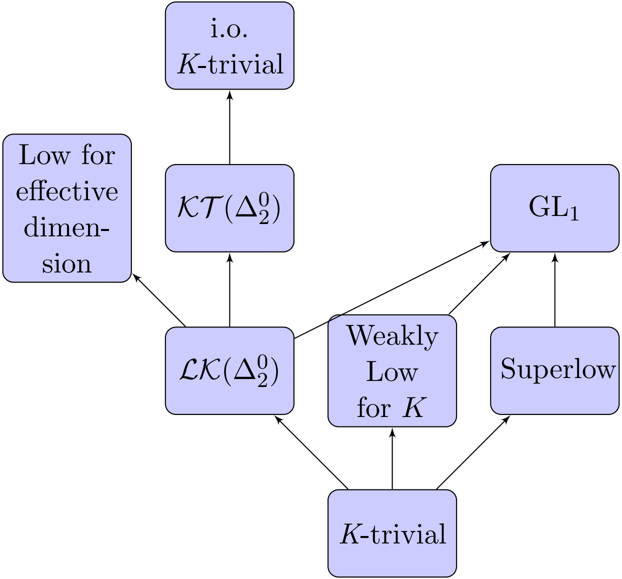

Weak Reducibilities

‘Strong’ reducibilities like $\leq_T$, $\leq_{tt}$, $\leq_m$ have an underlying map: $A \leq B$ iff $\exists \Phi : 2^\omega \to 2^\omega$ with $\Phi(B) = A$.

‘Weak’ reducibilities do not have such an underlying map. The examples we are concerned with all relate to Kolmogorov complexity.

Weak Reducibilities

Definition (Downey, Hirschfeldt, LaForte)

$A \leq_K B$ iff for all n, $K(A\restriction_n) \leq^+ K(B\restriction_n)$.

Definition (Nies)

$A \leq_{LK} B$ iff for all σ, $K^B(\sigma) \leq^+ K^A(\sigma)$.

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Lower $\leq_{LK}$-Cones

Since we no longer have an underlying map, uncountably many reals may be reducible to a single real under these reducibilities. So far, we know the $\leq_{LK}$-cone below A is:

- Countable, if A is low for K.
- Countable, if A is weakly low for K (Miller).
- Uncountable, if A is complete (Barmpalias, Lewis, Soskova).
- Uncountable, if A is non-GL_2 (Barmpalias, Lewis, Soskova).
- Uncountable, if A is Δ^0_2 but not low for K (Barmpalias).

Lower $\leq_{LK}$-Cones

Question

What can we say about lower $\leq_{LK}$-cones of reals in $\mathcal{LK}(\Delta^0_2)$?

Conjecture (Miller)

A has a countable lower $\leq_{LK}$-cone iff A is weakly low for K.

Theorem (Barmpalias, Vlek)

If A is i.o. K-trivial, then A has a countable lower $\leq_K$-cone.

Corollary

Every real in $\mathcal{KT}(\Delta^0_2)$ has a countable lower $\leq_K$-cone.

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Other Questions

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Every nonrecursive weakly low for K set is of hyperimmune degree (Miller, Nies). What about $\mathcal{LK}(\Delta^0_2)$*?*

Question

What can we say about the internal structures of $\mathcal{LK}(f)$ *and* $\mathcal{KT}(g)$ *for various* f *and* g *under* $\leq_{LK}$ *and* $\leq_K$*?*

Question

What about other lowness notions? C-triviality, lowness for C, etc?

Thanks!