Randomness for capacities

with applications to random closed sets

Jason Rute

Pennsylvania State University

Computability, Complexity, and Randomness
June 2014

Slides available at
www.personal.psu.edu/jmr71/

(Updated on June 17, 2014.)

Introduction and Motivation

The main idea

- All measures are additive:

$$\mu(A \cup B) = \mu(A) + \mu(B) \quad (A \text{ and } B \text{ disjoint})$$

- However there are also **nonadditive** "measures", in particular
  - subadditive "measures" (a.k.a. **capacities**)

$$\mu(A \cup B) \leqslant \mu(A) + \mu(B) \quad (A \text{ and } B \text{ disjoint})$$

  - superadditive "measures"

$$\mu(A \cup B) \geqslant \mu(A) + \mu(B) \quad (A \text{ and } B \text{ disjoint})$$

- Martin-Löf randomness can be extended to nonadditive "measures".
- This provides a **unified framework** for many notions in randomness.

Examples of randomness for capacities (subadditive)

- strong s-randomness (resp. strong f-randomness) . . . . (Reimann and many others)
  - = randomness on s-dimensional (resp. f-weighted) Hausdorff capacity
- s-energy randomness . . . . (Diamondstone/Kjos-Hanssen)
  - = randomness on s-dimensional Riesz capacity
- MLR for a class of measures . . . . (Bienvenu/Gács/Hoyrup/Rojas/Shen)
  - = randomness on the corresponding upper envelope capacity
- members of a MLR closed set
  - MLR closed sets . . . . (Barmpalias/Brodhead/Cenzer/Dashti/Weber)
  - zeros of MLR Brownian motion . . . . (Kjos-Hanssen/Nerode and A/B/S)
  - image of MLR n-dim. Brownian motion . . . . (Allen/Bienvenu/Slaman)
  - double points of MLR planar BM . . . . (Allen/Bienvenu/Slaman)
  - = randomness on the corresponding intersection capacity
- (Unfinished work) Lebesgue points of all computable Sobolev $W^{n,p}$ functions
  - = (some sort of) Schnorr randomness on n,p-Bessel capacity

More examples

- Randomness for superadditive measures loosely corresponds to
  - randomness for semimeasures (studied by Levin and Bienvenu/Hölzl/Porter/Shafer)
- Randomness for capacities can be used to characterize
  - effective dimension (studied by Lutz and everyone else)

An application: members of random closed sets

Random closed sets of Barmpalias/Brodhead/Cenzer/Dashti/Weber

- Construct a $\{0,1\}$-tree with no dead-ends:
  - Branch only left with probability 1/3.
  - Branch only right with probability 1/3.
  - Branch both directions with probability 1/3.
- A MLR tree is one constructed with a MLR in $3^{\mathbb{N}}$.
- A **BBCDW MLR closed set** is the set of paths through a MLR tree.
- What are the elements of a BBCDW MLR closed set?

Theorem (Diamondstone and Kjos-Hanssen)

- *If z is MLR on some probability measure μ satisfying the condition*

$$\iint |x-y|^{-\log_2(3/2)}\, d\mu(x) d\mu(y) < \infty,$$

- *then z is a member of some BBCDW MLR closed set.*

- D/K-H conjectured the converse holds. **I will show they were correct!**

Definitions and Basic Results

Nonadditive measures

Definition

A **regular nonadditive measure** on $2^{\mathbb{N}}$ or $\mathbb{R}^n$ is a set function C such that

- C is defined on all open and closed sets,
- C is finite on compact sets,
- $C(\varnothing) = 0$,
- C is monotone: $C(A) \leqslant C(B)$ for $A \subseteq B$,
- C continuous from below on open sets: $C(U_n) \uparrow C(U)$ if $U_n \uparrow U$,
- C continuous from above on compact sets: $C(K_n) \downarrow C(K)$ if $K_n \downarrow K$,
- C is outer regular: $C(A) = \inf\{C(U) : U \text{ open and } A \subseteq U\}$,
- C is inner regular: $C(A) = \sup\{C(K) : K \text{ compact and } K \subseteq A\}$.

Definition

If C is subadditive ($\mu(A \cup B) \geqslant \mu(A) + \mu(B)$) then call C a **capacity**.

Computable nonadditive measures

Definition

Say that a regular nonadditive measure is **computable** if

- C is uniformly lower-semicomputable on Σ^0_1 sets, and
- C is uniformly upper-semicomputable on Π^0_1 sets.

Or equivalently on $2^{\mathbb{N}}$,

- C is uniformly computable on clopen sets (not just cylinder sets).

Definition

A **Martin-Löf test** (U_n) is a uniform sequence of Σ^0_1 sets such that $C(U_n) \leqslant 2^{-n}$.
A point x is **Martin-Löf random** on C if $x \notin \bigcap_n U_n$ for all ML tests.

Remark

If C is countably subadditive, $C(\bigcup_n A_n) \leqslant \sum C(A_n)$,

- There is a universal test (same proof!).
- Can also use Solovay tests, integrable tests, etc.

An important lemma

For two capacities, write

$$C_1 =^{\times} C_2$$

if they are equal up to a multiplicative constant, i.e. there are two constants $0 < a < b$ such that for all A,

$$a C_1(A) \leqslant C_2(A) \leqslant b C_1(A).$$

Lemma

For two computable capacities, if $C_1 =^{\times} C_2$ then C_1 and C_2 have the same ML randoms.

Randomness for non-computable measures

- It is well known that MLR can be extended to noncomputable probability measures (Levin, Reimann/Slaman, Day/Miller, B/G/H/R/S)

Definition

A point $x \in X$ is **Martin-Löf random on μ** if x is not covered by any ML tests "computable from μ".

- Let $\mathcal{M}^+(X)$ denote the space of all finite Borel measures on X.

Lemma

If x is random on $\mu \in \mathcal{M}^+(X)$, then x is random on the prob. measure $\mu/\mu(X)$.

- Randomness can also be developed for non-computable capacities, but this is not needed in this talk.

Effective compactness

On any computable metric space X (e.g. $2^{\mathbb{N}}$, $\mathbb{R}^n$, $\mathcal{M}^+(2^{\mathbb{N}})$, $\mathcal{M}^+(\mathbb{R}^n)$) let $\mathcal{K}(X)$ denote the space of nonempty compact sets.

Definition

A nonempty compact set K is said to be[1] **computable in $\mathcal{K}(X)$** iff any of the following equivalent conditions hold:

1. K is computable in the Hausdorff metric on $\mathcal{K}(X)$.
2. The distance function $x \mapsto \mathrm{dist}(x,K)$ is computable.
3. K is the image of a total computable map $2^{\mathbb{N}} \to X$.
4. K is Π^0_1 and contains a computable dense subsequence.
5. (On $2^{\mathbb{N}}$) K is the set of paths of a computable tree with no dead branches.
6. (On $\mathbb{R}$) Both $\{a,b \in \mathbb{Q} : K \cap (a,b) \neq \varnothing\}$ and $\{a,b \in \mathbb{Q} : K \cap [a,b] = \varnothing\}$ are Σ^0_1.
7. The map $f \mapsto \max(K,f)$ is uniformly computable for continuous f.

[1] Others use the terms "located set", "computable set", or "decidable set". The term "effectively compact set" is usually reserved for a weaker notion.

Some Examples of Capacities and Characterizations of their Randoms Points

Upper envelopes / Randomness for classes of measures

- Let $\mathcal{C}$ be a compact subset of $\mathcal{M}^+(X)$ (finite Borel measures on X).
- The **upper envelope capacity** is defined as

$$\mathsf{Cap}_{\mathcal{C}}(A) = \sup_{\mu \in \mathcal{C}} \mu(A).$$

Theorem

If $\mathcal{C}$ computable in $\mathcal{K}(\mathcal{M}^+(X))$, then $\mathsf{Cap}_{\mathcal{C}}$ is a computable capacity.

Theorem (Basically Bienvenu-Hoyrup-Gács-Rojas-Shen)

If $\mathcal{C}$ computable in $\mathcal{K}(\mathcal{M}^+(X))$, the following are equivalent.

- *x is random on the upper envelope $\mathsf{Cap}_{\mathcal{C}}$.*
- *x is random on some measure $\mu \in \mathcal{C}$.*

Riesz capacity / Energy randomness

- s-energy of a measure $\mu \in \mathcal{M}^+(\mathbb{R}^n)$

$$\mathsf{Energy}_s(\mu) := \begin{cases} \iint |x-y|^{-s}\, d\mu(x) d\mu(y) & 0 < s \leqslant n \\ \iint \log \frac{1}{|x-y|}\, d\mu(x) d\mu(y) & s = 0 \end{cases}$$

- s-Riesz capacity (version 1)

$$\mathsf{C}_s(A) := \sup \left\{ \mu(A) \mid \mu \in \mathcal{M}^+(\mathbb{R}^n),\ \mathsf{Energy}_s(\mu) \leqslant 1 \right\}$$

Theorem

For computable s,

- $\{\mu \mid \mu \in \mathcal{M}^+(\mathbb{R}^n),\ \mathsf{Energy}_s(\mu) \leqslant 1\}$ *is a computable set in* $\mathcal{K}(\mathcal{M}^+(\mathbb{R}^n))$.
- C_s *is a computable (upper envelope) capacity*
- *The following are equivalent:*
  - x *is random on the Riesz capacity* C_s.
  - x *is random on some measure* $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ *such that* $\mathsf{Energy}_s(\mu) \leqslant 1$.
  - x *is random on some probability measure* μ *such that* $\mathsf{Energy}_s(\mu) < \infty$ $\Leftrightarrow$: ***s-energy random***.

Riesz capacity / Energy randomness

- s-Riesz capacity (version 1)

$$\mathsf{C}_s(A) := \sup\left\{\mu(A) \mid \mu \in \mathcal{M}^+(\mathbb{R}^n),\ \mathsf{Energy}_s(\mu) \leqslant 1\right\}.$$

- s-Riesz capacity (version 2)

$$\mathsf{Cap}_s(A) := \sup\left\{\frac{1}{\mathsf{Energy}_s(\mu)} \mid \mu \in \mathcal{M}_1(A)\right\}.$$

Theorem

$\mathsf{Cap}_s = (\mathsf{C}_s)^2$.

Corollary

The following are equivalent

- x is random on the Riesz capacity Cap_s.
- x is random on the Riesz capacity C_s.
- x is s-energy random.

Random compact sets / Members of MLR compact sets

- For a probabilist, a **random set** is a process which randomly generates a set, i.e. a set-valued random variable Z.
- Each random compact set corresponds to a probability measure $\mathbf{P}_Z$ on the space of compact sets $\mathcal{K}(2^{\mathbb{N}})$.
- A compact set K is a **MLR compact set** if it is MLR on $\mathbf{P}_Z$.

- The **intersection capacity** is defined as follows:

$$\mathsf{T}_Z(A) = \mathbf{P}\{Z \cap A \neq \varnothing\} = \mathbf{P}_Z\left\{K \in \mathcal{K}(2^{\mathbb{N}}) : K \cap A \neq \varnothing\right\}.$$

- $\mathsf{T}_Z(A)$ is the probability that A intersects some random compact set.

Theorem (R.)

For computable measure $\mathbf{P}_Z$ on $\mathcal{K}(2^{\mathbb{N}})$, the following are equivalent.

- *x is a random for the intersection capacity T_Z.*
- *x is a member of some $\mathbf{P}_Z$-MLR compact set.*

Applications of Potential Theory to Random Compact Sets

Proof of Diamondstone and Kjos-Hanssen's conjecture

Conjecture (Diamondstone/Kjos-Hanssen)

The following are equivalent.

- *x is a member of some BBCDW MLR compact set.*
- *x is $\log_2(3/2)$-energy random.*

- Let T_{BBCDW} be the intersection capacity of the BBCDW random set.
- Recall $\mathsf{Cap}_{\log_2(3/2)}$ denotes the $\log_2(3/2)$-Riesz capacity.

Equivalent Conjecture

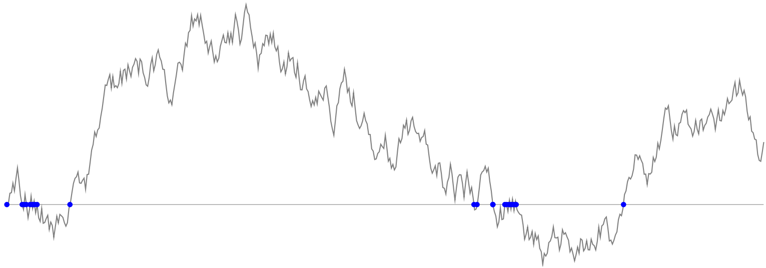
T_{BBCDW} and $\mathsf{Cap}_{\log_2(3/2)}$ have the same randoms.

The conjecture follows from...

Theorem (Lyons)

$\mathsf{T}_{BBCDW} =^{\times} \mathsf{Cap}_{\log_2(3/2)}$.

Zeros of Brownian motion



The zeros of Brownian motion B form a random closed set.

$$Z_B = \{t \in [a,b] : B(t) = 0\}$$

Question (Allen/Bienvenu/Slaman)

Which reals are zeros of some MLR Brownian motion?

Zeros of a MLR Brownian motion

Let T_{Z_B} be the intersection capacity of the random closed set Z_B of zeros.

$$\mathsf{T}_{Z_B}(A) := \mathbb{P}\{Z_B \cap A \neq \varnothing\}$$

Theorem (Kakutani)

$\mathsf{T}_{Z_B} =^{\times} \mathsf{Cap}_{1/2}$ when restricted to an interval $[a,b]$ $(0 < a < b)$.

Corollary

T_{Z_B} and $\mathsf{Cap}_{1/2}$ have the same randoms (in $[a,b]$)

Theorem (R.—Allen/Bienvenu/Slaman were very close!)

The following are equivalent.

- *x is a zero time of some MLR Brownian motion.*
- *x is $1/2$-energy random.*

Image of an n-dimensional Brownian motion

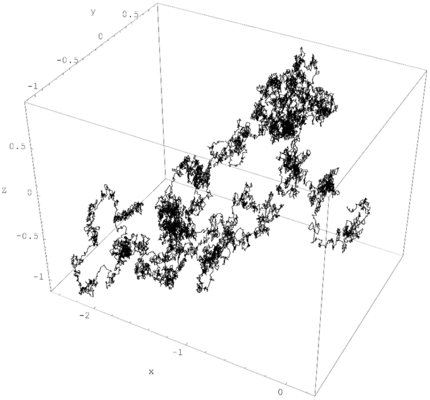


Figure : Image of a 3D Brownian motion (http://en.wikipedia.org/wiki/Wiener_process)

The image of an n-dimensional Brownian motion forms a random closed set.

$$B([a,b]) = \{B(t) : a \leqslant t \leqslant b\}$$

Question (Allen/Bienvenu/Slaman)

Which points are in the image of some n-dimensional MLR Brownian motion?

Image of a MLR n-dimension Brownian motion

Let be the intersection capacity of $B([a,b])$.

$$\mathsf{T}_{B([a,b])}(A) := \mathbb{P}\{B([a,b]) \cap A \neq \varnothing\}$$

Theorem (Kakutani)

$\mathsf{T}_{B([a,b])} =^{\times} \mathsf{Cap}_{n-2}$ when $n \geqslant 2$.

Corollary

$\mathsf{T}_{B([a,b])}$ and Cap_{n-2} have the same randoms (for $n \geqslant 2$).

Theorem (R.)

The following are equivalent for $x \in \mathbb{R}^n$ ($x \neq 0$) and $n \geqslant 2$.

- *x is in the image of some n-dimensional MLR Brownian motion.*
- *x is $(n-2)$-energy random.*

Double points of planar Brownian motion

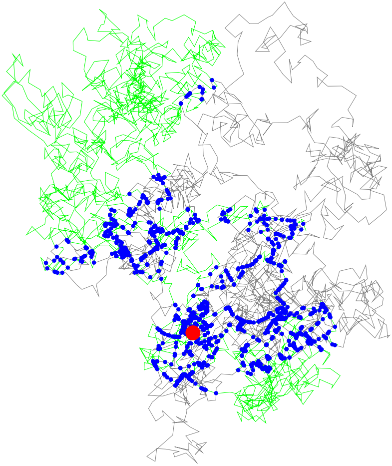


Figure : Intersection (in blue) of two planar Brownian motions starting at the origin.

- The double points of a planar Brownian motion are the points where it intersects itself.

Question (Allen/Bienvenu/Slaman)

Which points are the double points of some MLR planar Brownian motion?

- Closely related, consider the intersection points of two independent planar Brownian motions which start at the origin.
- This forms a random closed set.

Double points of MLR planar BM

- Let $\mathsf{T}_{B_1 \cap B_2}$ be the intersection capacity of random closed set $B_1([a,b]) \cap B_2([a,b])$.

$$\mathsf{T}_{B_1 \cap B_2}(A) := \mathbb{P}\{B_1([a,b]) \cap B_2([a,b]) \cap A \neq \varnothing\}$$

- Define $\mathsf{Cap}_{\log^2}$ and $\log^2$**-energy random** using the energy

$$\iint (\log|x-y|)^2 \, d\mu(x) d\mu(y).$$

Theorem (Fitzsimmons/Salisbury, also Permantal/Peres)

$\mathsf{T}_{B_1 \cap B_2} =^{\times} \mathsf{Cap}_{\log^2}$.

Theorem (R.)

The following are equivalent for $x \in \mathbb{R}^2$ ($x \neq 0$).

- *x is a double point of some MLR planar Brownian motion.*
- *x is in the intersection of some pair of independent MLR planar BM.*
- *x is $\log^2$-energy random.*

Capacities and Effective Hausdorff Dimension

Hausdorff capacity vs Hausdorff measure

- ***s*-Hausdorff capacity** (a.k.a. ***s*-Hausdorff content**) of radius $r \in (0, \infty]$.

$$\mathcal{H}^s_r(A) := \inf \sum_i 2^{-s|\sigma_i|}$$

  where the infimum is over covers $\bigcup_i [\sigma_i] \supseteq A$ such that $2^{|\sigma_i|} \leqslant r$.
- s-dimensional **Hausdorff measure**

$$\mathcal{H}^s(A) := \lim_{r \to 0} \mathcal{H}^s_r(A)$$

- Same null sets:

$$\mathcal{H}^s(A) = 0 \quad \Leftrightarrow \quad \mathcal{H}^s_r(A) = 0.$$

- $\mathcal{H}^s$ is an ugly measure (open sets have infinite measure!)
- $\mathcal{H}^s_r$ is a nice capacity.
- $\mathcal{H}^s_\infty$ is called **vehement weight** by some in algorithmic randomness

Frostman's Lemma / Strong s-randomness

Frostman's Lemma

The following are equivalent.

- $\mathcal{H}^s_\infty(A) > 0$
- $\mu(A) > 0$ for some prob. measure μ such that $\mu(\sigma) \leqslant^\times 2^{-s|\sigma|}$ for all σ.

- A point $x \in 2^\mathbb{N}$ is **strongly s-random** if $KM(x \upharpoonright n) \geqslant sn + O(1)$. ($KM$ is a priori complexity.)
- A point $x \in 2^\mathbb{N}$ is **vehemently s-random** if x is $\mathcal{H}^s_\infty$-random.
- A point $x \in 2^\mathbb{N}$ is **s-capacitable** if x is μ-random for some prob. measure μ s.t. $\mu(\sigma) \leqslant^\times 2^{-s|\sigma|}$ for all σ.

Effective Frostman's Lemma (Reimann, Kjos-Hanssen)

The following are equivalent.

- x is strong s-random.
- x is s-vehemently random ($\mathcal{H}^s_\infty$-random).
- x is s-capacitable.

Effective dimension (Frostman's Theorem)

Theorem (Frostman's Theorem)

$$\dim(A) = \sup\{s \mid \mathcal{H}^s_r(A) > 0\} = \sup\{s \mid \mathsf{Cap}_s(A) > 0\}$$

Theorem (Effective Frostman's Theorem (Reimann, Diamondstone/K-H))

$$\mathrm{cdim}(x) = \sup\{s \mid x \textit{ is strongly } s\textit{-random}\} = \sup\{s \mid x \textit{ is } s\textit{-energy random}\}$$

Question (Reimann)

Are strong s-randomness and s-energy randomness the same?

Answer

No.

Strong s-randomness vs. s-energy randomness

Question

Are strong s-randomness and s-energy randomness equal?

Theorem (Maz'ya/Khavin, also in Adams/Hedberg book)

There is a(n effective) closed set E such that $\mathcal{H}^s_r(E) > 0$, but $\mathsf{Cap}_s(E) = 0$.

Corollary (R.)

There is an x which is strongly s-random, but not s-energy random.

Summary of known results

Known Results (No arrows reverse)

$\operatorname{cdim} x > s$

$\Rightarrow$ x is s-energy random $\Leftrightarrow$ x is random on Cap_s

$\Rightarrow$ x is strongly s-random $\Leftrightarrow$ x is random on $\mathcal{H}^s_r$

$\Rightarrow$ x is weakly s-random (that is $K(x \upharpoonright n) \geqslant sn + O(1)$)

$\Rightarrow$ $\operatorname{cdim} x \geqslant s$.

Remark

Weak s-randomness is not associated (at least in any nice way) with a capacity.

Cylinder sets do not determine randomness

- Let T_{BBCDW} be the intersection capacity of the BBCDW random sets.
- Let $\mathcal{H}_\infty^{\log_2(3/2)}$ be the Hausdorff capacity for dimension $s = \log_2(3/2)$.
- It turns out that
    - $\mathsf{T}_{BBCDW}([\sigma]) = \mathcal{H}_\infty^{\log_2(3/2)}([\sigma]) = (2/3)^{|\sigma|}$.
    - T_{BBCDW}-random ($\log_2(3/2)$-energy random) $\neq \mathcal{H}_\infty^{\log_2(3/2)}$-random (strong $\log_2(3/2)$-random).

Remark

Therefore, the values of a capacity **on cylinder sets** is not enough to determine its randomness.

Superadditive Measures and Semi-measures

Semi-measures

- Semi-measures:
$$\rho(\sigma 0) + \rho(\sigma 1) \leqslant \rho(\sigma).$$
- Levin and Bienvenu/Hölzl/Porter/Shafer have looked at randomness for computable and left c.e. semimeasures.
- It is messy in the left c.e. case!

- Can extend ρ to the smallest superadditive measure $C \geqslant \rho$.
- **Blind (Hippocratic) randomness** is ML randomness (for non-computable measures) which doesn't use the measure as an oracle.

Theorem (R.)

If ρ is computable (resp. left c.e.), the following are equivalent.

- *x is random (resp. blind random) on C.*
- *x is random on ρ using Bienvenu/Hölzl/Porter/Shafer definitions.*
- *x is blind μ-random for every probability measure $\mu \geqslant \rho$.*

There is so much more to say...

...but I have run out of time.

Summary

Randomness for nonadditive measures...

- provides a unified framework for concepts in randomness
- helps us to prove new results
- simplifies the proofs of old results
- gives us 60-years-worth of theorems in capacity theory to draw from!

Closing Thoughts

Thank You!

These slides will be available on my webpage:

http://www.personal.psu.edu/jmr71/

Or just Google™ me, "Jason Rute".