Schnorr randomness versions of K, C, LR, vL-reducibilities

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Motivation

- Give Schnorr randomness versions of theorems for ML-randomness.

- Why important or interesting?
  - Deep understanding of the theorems.
  - Unexpected findings.
Question 1

Question 1.
ML-randomness has characterizations by K and C. Schnorr randomness by K_M where M is a computable measure machine. What is a Schnorr randomness version of the one by C?

Answer.
Schnorr randomness has a characterization by C_M where M is a total machine.
Question 2

Question 2.
Computably-traceable-reducibility can be characterized by relative Schnorr randomness?
(Problem 8.4.22 in Nies’ book)

Answer.
Yes.
Question 3.
2-randomness can be characterized by infinitely-often maximality of complexity.
Is there a Schnorr randomness version?

Partial answer.
No.
Schnorr version of C
ML-randomness

**Definition** (Martin-Löf 1966)

$X \in 2^\omega$ is ML-random if $x \notin \bigcap_n U_n$ for each ML-test $\{U_n\}$, i.e., $U_n$ is a uniformly c.e. open set with $\mu(U_n) \leq 2^{-n}$.

**Theorem**

$X$ is ML-random iff $K(X \mid n) > n - O(1)$. (Levin, Schnorr 1973)

$X$ is ML-random iff $C(X \mid n) > n - K(n) - O(1)$. (Miller-Yu 2008)
Schnorr randomness

**Definition** (Schnorr 1971)

\[ X \in 2^\omega \text{ is Schnorr random if } x \notin \bigcap_n U_n \text{ for each Schnorr test } \{U_n\}, \text{ i.e., } \{U_n\} \text{ is a ML-test and } \mu(U_n) \text{ is uniformly computable.} \]

**Theorem** (Downey-Griffiths 2004)

\[ X \in 2^\omega \text{ is Schnorr random iff } K_M(X \upharpoonright n) > n - O(1) \text{ for every computable measure machine } M, \text{ i.e., } M \text{ is a prefix-free machine and } \sum_{\sigma \in \text{dom}(M)} 2^{-|\sigma|} \text{ is computable.} \]
Schnorr version of C

Theorem (M.)

$X$ is Schnorr random iff, for every computable measure machine $M$ and every total machine $N$, we have

$$C_N(X \mid n) > n - K_M(n) - O(1).$$
Related results

**Theorem** (Bienvenu-Merkle 2007)

$X$ is Schnorr random iff, for every decidable prefix-free machine $M$ and every computable order $g$, we have

$$K_M(X \upharpoonright n) > n - g(n) - O(1).$$

**Theorem** (Hölzl-Merkle 2010)

$A$ is Schnorr trivial iff, for every computable order $g$, there exists a total machine $M$ such that

$$K_M(A \upharpoonright g(n)) \leq n + O(1).$$
Schnorr version of \( C \)-reducibility

\[ X \leq_C Y \text{ if} \]

\[ C(X \upharpoonright n) \leq C(Y \upharpoonright n) + O(1). \]

**Definition**

\( X \leq_{tm} Y \) if, for every total machine \( M \), there exists a total machine \( N \) such that

\[ C_N(X \upharpoonright n) \leq C_M(Y \upharpoonright n) + O(1). \]

We come back to this notion later.
Schnorr version of LR
LR-reducibility

A is low for MLR if every A-ML-random set is ML-random. A is low for K if \( K(n) \leq K^A(n) + O(1) \). These notions are equivalent.

Theorem (Kjos-Hanssen-Miller-Solomon 2012)
The following are equivalent for \( X, Y \in 2^\omega \):

(i) Every \( X \)-ML-random set is \( Y \)-ML-random. (\( X \leq_{LR} Y \))

(ii) \( K^Y(n) \leq K^X(n) + O(1) \). (\( X \leq_{LK} Y \))
Schnorr version of LR?

The following are equivalent for $A \in 2^\omega$:

(i) $A$ is low for Schnorr Randomness.
(ii) $A$ is computably traceable.
(iii) $A$ is low for computable measure machines.


Nies (Problem 8.4.22 in his book) asked whether the reducibility version of the equivalence between (i) and (ii) holds.
Definition (Nies)

\( A \leq_{CT} B \) if there is a computable order \( h \) such that for each \( f \leq_T A \) there exists \( p \leq_T B \) such that \( f(n) \in D_{p(n)} \) and \( |D_{p(n)}| \leq h(n) \) for every \( n \).

Theorem (M.)

The following are equivalent for \( A, B \in 2^\omega \):

(i) \( A \leq_{CT} B \).

(ii) Every Schnorr random set relative to \( B \) is Schnorr random relative to \( A \).
Techniques

- low for tests = low for random (open covering method)
  by Bienvenu-Miller 2012

- LR = LK
  by Kjos-Hanssen-Miller-Solomon 2012

- low for Schnorr tests = low for c.m.m.
  by Bienvenu in arXiv.

- Combine and relativize them, then you get the result!
Schnorr version of $vL$
K-reducibility

Definition

\[ X \leq_K Y \text{ if } K(X \upharpoonright n) \leq K(Y \upharpoonright n) + O(1). \]
vL-reducibility

**Definition** (Miller-Yu 2008)

\[ X \leq_{vL} Y \text{ if, for every } Z, \]

\[ X \oplus Z \in \text{MLR} \Rightarrow Y \oplus Z \in \text{MLR}. \]

For \( X, Y \in \text{MLR}, \)

\[ X \leq_{vL} Y \iff Y \leq_{LR} X. \]
C,K implies vL

**Theorem** (Miller-Yu 2008)

(i) \( X \leq_K Y \) implies \( X \leq_{vL} Y \).

(ii) \( X \leq_C Y \) implies \( X \leq_{vL} Y \).
Schnorr versions

**Definition** (Downey-Griffiths 2004) 

\( X \leq_{Sch} Y \) if, for every c.m.m. \( M \), there exists a c.m.m. \( N \) such that 

\[
K_N(X \upharpoonright n) \leq K_M(X \upharpoonright n) + O(1).
\]

**Theorem** (M. 2011, M.-Rute 2013) 

\( X \oplus Y \) is Schnorr random iff \( X \) is Schnorr random and \( Y \) is Schnorr random uniformly relative to \( X \).
Definition

\( X \leq_{vLS} Y \) if, for every \( Z \),

\[
X \oplus Z \in \text{SR} \Rightarrow Y \oplus Z \in \text{SR}.
\]
Schnorr versions hold

Theorem (M.)

(i) $X \leq_{Sch} Y$ implies $X \leq_{vLS} Y$.

(ii) $X \leq_{tm} Y$ implies $X \leq_{vLS} Y$. 

.
Technique

- An extension of Ample Excess Lemma
Theorem (Ample Excess Lemma; Miller-Yu 2008)

(i) $X$ is ML-random iff $\sum_n 2^{n - K(X \upharpoonright n)} < \infty$.

(ii) If $X$ is ML-random, then

$$K(X \upharpoonright n) \geq n + K^X(n) - O(1).$$
An extension of AEL

Observation

For a machine $M$, we define a function $f_M : 2^\omega \to \mathbb{R}$ by

$$f_M(X) = \sum_{n=0}^{\infty} 2^{n-K_M(X \upharpoonright n)}.$$ 

Then, we have

$$\int f_M(X) d\mu = \hat{\Omega}_M = \sum \{2^{-K_M(\sigma)} : \sigma \in 2^{<\omega}, K_M(\sigma) < \infty \}.$$
If $U$ is a universal prefix-free machine, then

$$\Omega_U = \sum_{\sigma \in \text{dom}(U)} 2^{-|\sigma|}$$

and $\hat{\Omega}_U$ are ML-random.

If $M$ is a computable measure machine, then $\Omega_M$ and $\hat{\Omega}_M$ are computable.

In general, we have

$$\hat{\Omega}_M \leq_S \Omega_M$$

where $\leq_S$ is Solovay reducibility. The converse does not hold in general.
Corollary

$X$ is Schnorr random iff $\sum_n 2^{n-K_M(X|n)} < \infty$ for every computable measure machine $M$. 
Recall that

\[ f_M(X) = \sum_{n=0}^{\infty} 2^{n-K_M(X|n)}. \]

Then

\[
\int f_M(X) \, d\mu = \int \sum_{n=0}^{\infty} 2^{n-K_M(X|n)} \, d\mu
\]

\[
= \sum_{n=0}^{\infty} \sum_{\sigma \in 2^n} 2^{n-K_M(\sigma)} \cdot 2^{-n}
\]

\[
= \sum_{\sigma \in 2^{<\omega}} 2^{-K_M(\sigma)}
\]

\[
= \hat{\Omega}_M
\]
Proposition (M.)
Let $X$ be a Schnorr random set. For every computable measure machine $M$, there exists a uniformly computable measure machine $N$ such that

$$K_M(X \upharpoonright n) \geq n + K_{N\cdot X}(n) - O(1).$$
Theorem (Miller 2009)

$X$ is 2-random if and only if

$$K(X \upharpoonright n) \geq n + K(n) - O(1)$$

for infinitely many $n$.

(i) Ample Excess Lemma

(ii) $\Omega$ is $X$-ML-random (low for $\Omega$) iff $K(n) \leq K^X(n)$ for infinitely many $n$ (weakly low for $K$)
Definition
A set $A$ is **weakly low for c.m.m.** if, for every u.c.m.m. $M$, there exists a c.m.m. $N$, such that

$$K_N(n) \leq K_{MA}(n) + O(1)$$

for infinitely many $n$.

Proposition
Every set is weakly low for c.m.m.
Theorem (M.)

For a c.m.m. $M$, there exists a c.m.m. $N$ such that, for every Schnorr random set $X$,

$$K_M(X \upharpoonright n) \geq n + K_N(n) - O(1)$$

for infinitely many $n$. 
Open question

Is the following notion equivalent to Schnorr randomness?

For every total machine $M$,

$$C_M(X \mid n) \geq n - O(1)$$

for infinitely many $n$. 
Summary

- We looked at Schnorr-randomness versions of some theorems on ML-randomness.

- Because of non-universality, we should take care about the dependency on the machine, which (may) deepen the understanding.

- Hierarchy of Schnorr randomness?
Thank you for listening.