Finite automorphism bases for degree structures

Mariya I. Soskova¹

Sofia University

joint work with Theodore Slaman

¹Supported by a Marie Curie International Outgoing Fellowship STRIDE (298471), Sofia University Science Fund and BNSF Grant No. DMU
03/07/12.12.2011
Automorphism bases

**Definition**

Let $\mathcal{A}$ be a structure with domain $A$. A set $B \subseteq A$ is an automorphism base for $\mathcal{A}$ if whenever $f$ and $g$ are automorphisms of $\mathcal{A}$, such that $(\forall x \in B)(f(x) = g(x))$, then $f = g$.
Automorphism bases

**Definition**

Let $\mathcal{A}$ be a structure with domain $A$. A set $B \subseteq A$ is an automorphism base for $\mathcal{A}$ if whenever $f$ and $g$ are automorphisms of $\mathcal{A}$, such that $(\forall x \in B)(f(x) = g(x))$, then $f = g$.

Equivalently if $f$ is an automorphism of $\mathcal{A}$ and $(\forall x \in B)(f(x) = x)$ then $f$ is the identity.
Automorphism bases

**Definition**
Let $\mathcal{A}$ be a structure with domain $A$. A set $B \subseteq A$ is an automorphism base for $\mathcal{A}$ if whenever $f$ and $g$ are automorphisms of $\mathcal{A}$, such that $(\forall x \in B)(f(x) = g(x))$, then $f = g$.

Equivalently if $f$ is an automorphism of $\mathcal{A}$ and $(\forall x \in B)(f(x) = x)$ then $f$ is the identity.

**Theorem (Slaman and Woodin)**

*There is an element $g \leq 0^{(5)}$ such that $\{g\}$ is an automorphism base for the structure of the Turing degrees $\mathcal{D}_T$.***
Automorphism bases

Definition

Let $\mathcal{A}$ be a structure with domain $A$. A set $B \subseteq A$ is an automorphism base for $\mathcal{A}$ if whenever $f$ and $g$ are automorphisms of $\mathcal{A}$, such that $(\forall x \in B)(f(x) = g(x))$, then $f = g$.

Equivalently if $f$ is an automorphism of $\mathcal{A}$ and $(\forall x \in B)(f(x) = x)$ then $f$ is the identity.

Theorem (Slaman and Woodin)

There is an element $g \leq 0^{(5)}$ such that $\{g\}$ is an automorphism base for the structure of the Turing degrees $\mathcal{D}_T$.

$\text{Aut}(\mathcal{D}_T)$ is countable and every member has an arithmetically definable presentation.
Part I: The local structure of the Turing degrees

Definition

A set of degrees $Z$ contained in $D_T(\leq 0')$ is uniformly low if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$ representing the degrees in $Z$, and a computable function $f$ such that $\{f(i)\}_{\emptyset'}$ is the Turing jump of $\bigoplus_{j<i}Z_j$.

Example: If $\bigoplus_{i<\omega}A_i$ is low then $A = \{d_T(A_i) | i<\omega\}$ is uniformly low.

Theorem (Slaman and Woodin)

If $Z$ is a uniformly low subset of $D_T(\leq 0')$ then $Z$ is definable from parameters in $D_T(\leq 0')$. 
Part I: The local structure of the Turing degrees

Definition

A set of degrees $\mathcal{Z}$ contained in $\mathcal{D}_T(\leq 0')$ is uniformly low if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$, representing the degrees in $\mathcal{Z}$, and a computable function $f$ such that $\{f(i)\}^{\emptyset'}$ is the Turing jump of $\bigoplus_{j<i} Z_j$.

Example:

If $\bigoplus_{i<\omega} A_i$ is low then $A = \{d_T(A_i) | i<\omega\}$ is uniformly low.

Theorem (Slaman and Woodin)

If $\mathcal{Z}$ is a uniformly low subset of $\mathcal{D}_T(\leq 0')$ then $\mathcal{Z}$ is definable from parameters in $\mathcal{D}_T(\leq 0')$. 
Part I: The local structure of the Turing degrees

Definition

A set of degrees \( \mathcal{Z} \) contained in \( D_T(\leq 0') \) is uniformly low if it is bounded by a low degree and there is a sequence \( \{Z_i\}_{i<\omega} \), representing the degrees in \( \mathcal{Z} \), and a computable function \( f \) such that \( \{f(i)\}^{\emptyset'} \) is the Turing jump of \( \bigoplus_{j<i} Z_j \).

Example: If \( \bigoplus_{i<\omega} A_i \) is low then \( \mathcal{A} = \{d_T(A_i) \mid i < \omega\} \) is uniformly low.
Part I: The local structure of the Turing degrees

Definition

A set of degrees $Z$ contained in $\mathcal{D}_T(\leq 0')$ is uniformly low if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$, representing the degrees in $Z$, and a computable function $f$ such that $\{f(i)\}^{0'}$ is the Turing jump of $\bigoplus_{j<i} Z_j$.

Example: If $\bigoplus_{i<\omega} A_i$ is low then $A = \{d_T(A_i) \mid i < \omega\}$ is uniformly low.

Theorem (Slaman and Woodin)

If $Z$ is a uniformly low subset of $\mathcal{D}_T(\leq 0')$ then $Z$ is definable from parameters in $\mathcal{D}_T(\leq 0')$. 
Applications of the coding theorem

1. Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^\mathcal{M}, 0^\mathcal{M}, +^\mathcal{M}, \times^\mathcal{M}, \leq^\mathcal{M})$. 

Mariya I. Soskova (Sofia University)
Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^\mathcal{M}, 0^\mathcal{M}, +^\mathcal{M}, \times^\mathcal{M}, \leq^\mathcal{M})$.

If $\mathcal{Z} \subseteq \mathcal{D}_T(\leq \mathbf{0}')$ is uniformly low and represented by the sequence $\{Z_i\}_{i < \omega}$ then there are $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and a function $\varphi : \mathbb{N}^\mathcal{M} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\varphi(i^\mathcal{M}) = d_T(Z_i)$. 
Applications of the coding theorem

1. Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^\mathcal{M}, 0^\mathcal{M}, +^\mathcal{M}, \times^\mathcal{M}, \leq^\mathcal{M})$.

2. If $\mathcal{Z} \subseteq \mathcal{D}_T(\leq 0')$ is uniformly low and represented by the sequence $\{Z_i\}_{i<\omega}$ then there are $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and a function $\varphi : \mathbb{N}^\mathcal{M} \rightarrow \mathcal{D}_T(\leq 0')$ such that $\varphi(i^\mathcal{M}) = d_T(Z_i)$.

We call such a function an indexing of $\mathcal{Z}$. 

Using parameters we can define the set of c.e. degrees:

\[ K = \bigsqcup_{e < \omega} W_e. \]

By Sacks' Splitting theorem there are low disjoint c.e. sets \( A \) and \( B \) such that \( K = A \cup B \).

Represent \( A \) and \( B \) as \( \bigsqcup_{e < \omega} A_e \) and \( \bigsqcup_{e < \omega} B_e \).

Note that \( W_e \) is the disjoint union of \( A_e \) and \( B_e \).

The set \( A = \{ d_T(A_e) | e < \omega \} \) and \( B = \{ d_T(B_e) | e < \omega \} \) are uniformly low and hence definable with parameters.

A degree \( x \) is c.e. if it is the join of an element from \( A \) and an element from \( B \).
Using parameters we can define the set of c.e. degrees:
Consider the set \( K = \bigoplus_{e<\omega} W_e \).
Applications of the coding theorem

Using parameters we can define the set of c.e. degrees:
Consider the set $K = \bigoplus_{e<\omega} W_e$. By Sacks’ Splitting theorem there are low disjoint c.e. sets $A$ and $B$ such that $K = A \cup B$. 
Using parameters we can define the set of c.e. degrees: Consider the set $K = \bigoplus_{e<\omega} W_e$. By Sacks’ Splitting theorem there are low disjoint c.e. sets $A$ and $B$ such that $K = A \cup B$. Represent $A$ and $B$ as $\bigoplus_{e<\omega} A_e$ and $\bigoplus_{e<\omega} B_e$. 
Using parameters we can define the set of c.e. degrees: Consider the set $K = \bigoplus_{e<\omega} W_e$. By Sacks’ Splitting theorem there are low disjoint c.e. sets $A$ and $B$ such that $K = A \cup B$.

Represent $A$ and $B$ as $\bigoplus_{e<\omega} A_e$ and $\bigoplus_{e<\omega} B_e$. Note that $W_e$ is the disjoint union of $A_e$ and $B_e$. 
Using parameters we can define the set of c.e. degrees: Consider the set $K = \bigoplus_{e<\omega} W_e$. By Sacks’ Splitting theorem there are low disjoint c.e. sets $A$ and $B$ such that $K = A \cup B$.

Represent $A$ and $B$ as $\bigoplus_{e<\omega} A_e$ and $\bigoplus_{e<\omega} B_e$. Note that $W_e$ is the disjoint union of $A_e$ and $B_e$.

The set $A = \{d_T(A_e) \mid e < \omega\}$ and $B = \{d_T(B_e) \mid e < \omega\}$ are uniformly low and hence definable with parameters.
Using parameters we can define the set of c.e. degrees: Consider the set $K = \bigoplus_{e < \omega} W_e$. By Sacks’ Splitting theorem there are low disjoint c.e. sets $A$ and $B$ such that $K = A \cup B$.

Represent $A$ and $B$ as $\bigoplus_{e < \omega} A_e$ and $\bigoplus_{e < \omega} B_e$. Note that $W_e$ is the disjoint union of $A_e$ and $B_e$.

The set $A = \{ d_T(A_e) \mid e < \omega \}$ and $B = \{ d_T(B_e) \mid e < \omega \}$ are uniformly low and hence definable with parameters.

A degree $x$ is c.e. if it is the join of an element from $A$ and an element from $B$. 
The goal

**Theorem (Slaman and Woodin)**

*There are finitely many $\Delta^0_2$ parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^\mathcal{M} \to D_T(\leq 0')$ such that $\psi(e^\mathcal{M}) = d_T(W_e)$.***
The goal

Theorem (Slaman and Woodin)

There are finitely many $\Delta^0_2$ parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq 0')$ such that $\psi(e^\mathcal{M}) = d_T(W_e)$.

Note that if we have an automorphism $\pi$ of $\mathcal{D}_T(\leq 0')$ which fixes these parameters then $\pi$ fixes every c.e degree.
The goal

Theorem (Slaman and Woodin)

There are finitely many \( \Delta^0_2 \) parameters which code a model of arithmetic \( \mathcal{M} \) and an indexing of the c.e. degrees: a function \( \psi : \mathbb{N}^\mathcal{M} \rightarrow \mathcal{D}_T(\leq 0') \) such that \( \psi(e^\mathcal{M}) = d_T(W_e) \).

Note that if we have an automorphism \( \pi \) of \( \mathcal{D}_T(\leq 0') \) which fixes these parameters then \( \pi \) fixes every c.e degree.

The Goal

Extend this result to find finitely many \( \Delta^0_2 \) parameters that code a model of arithmetic \( \mathcal{M} \) and an indexing \( \varphi \) of the \( \Delta^0_2 \) Turing degrees.
The goal

Theorem (Slaman and Woodin)

There are finitely many $\Delta^0_2$ parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^\mathcal{M} \to \mathcal{D}_T(\leq 0')$ such that $\psi(e^\mathcal{M}) = d_T(W_e)$.

Note that if we have an automorphism $\pi$ of $\mathcal{D}_T(\leq 0')$ which fixes these parameters then $\pi$ fixes every c.e degree.

The Goal

Extend this result to find finitely many $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing $\varphi$ of the $\Delta^0_2$ Turing degrees.

We will call $e$ an index for a $\Delta^0_2$ set $X$ if $\{e\}^{0'}$ is the characteristic function of $X$. 
Step 1: Reducing to low sets

Lemma

If \( x \leq_T 0' \) then there are low degrees \( g_1, g_2, g_3, g_4 \), such that 
\[
x = (g_1 \lor g_2) \land (g_3 \lor g_4).
\]
Step 1: Reducing to low sets

Lemma

If $x \leq_{T} 0'$ then there are low degrees $g_1, g_2, g_3, g_4$, such that $x = (g_1 \lor g_2) \land (g_3 \lor g_4)$.

- Suppose that we know how to map an index $e^M$ of a low $\Delta^0_2$ set $G$ to the degree $\varphi(e^M) = d_T(G)$. 
Step 1: Reducing to low sets

Lemma

If $x \leq T 0'$ then there are low degrees $g_1, g_2, g_3, g_4$, such that
$x = (g_1 \lor g_2) \land (g_3 \lor g_4)$.

- Suppose that we know how to map an index $e^M$ of a low $\Delta_2^0$ set $G$ to the degree $\varphi(e^M) = d_T(G)$.
- If in $M$ “$e$ is an index of a non-low $\Delta_2^0$ set $X$” then we search in $M$ for indices $e_1, e_2, e_3, e_4$ of low $\Delta_2^0$ sets which define the degree of $X$. 

Mariya I. Soskova (Sofia University)
Step 1: Reducing to low sets

**Lemma**

*If* \( x \leq_{T} 0' \) *then there are low degrees* \( g_1, g_2, g_3, g_4 \), *such that*

\[
x = (g_1 \lor g_2) \land (g_3 \lor g_4).
\]

- Suppose that we know how to map an index \( e^M \) of a low \( \Delta^0_2 \) set \( G \) to the degree \( \varphi(e^M) = d_T(G) \).
- If in \( M \) “\( e \) is an index of a non-low \( \Delta^0_2 \) set \( X \)” then we search in \( M \) for indices \( e_1, e_2, e_3, e_4 \) of low \( \Delta^0_2 \) sets which define the degree of \( X \).
- We map \( e^M \) to \((\varphi(e_1^M) \lor \varphi(e_2^M)) \land (\varphi(e_3^M) \lor \varphi(e_4^M))\).
Step 2: Distinguishing between low $\Delta^0_2$ sets

Theorem

There exists a uniformly low set of Turing degrees $\mathcal{Z}$, such that every low Turing degree $x$ is uniquely positioned with respect to the c.e. degrees and the elements of $\mathcal{Z}$. 

1. $b_i$ and $c_i$ are elements of $\mathcal{Z}$.
2. $g_i$ is the least element below $a_i$ which joins $b_i$ above $c_i$.
3. $x \leq g_1 \lor g_2$.
4. $y \nleq g_1 \lor g_2$. 

Mariya I. Soskova  ( Sofia University )
Step 2: Distinguishing between low $\Delta^0_2$ sets

Theorem

There exists a uniformly low set of Turing degrees $\mathcal{Z}$, such that every low Turing degree $x$ is uniquely positioned with respect to the c.e. degrees and the elements of $\mathcal{Z}$.

If $x, y \leq 0', x' = 0'$ and $y \nleq x$ then there are $g_i \leq 0'$, c.e. degrees $a_i$ and $\Delta^0_2$ degrees $c_i, b_i$ for $i = 1, 2$ such that:

1. $b_i$ and $c_i$ are elements of $\mathcal{Z}$.
2. $g_i$ is the least element below $a_i$ which joins $b_i$ above $c_i$.
3. $x \leq g_1 \lor g_2$.
4. $y \nleq g_1 \lor g_2$. 
Applications

Theorem (Biinterpretability with parameters)

There are finitely many $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the $\Delta^0_2$ degrees.
Applications

Theorem (Biinterpretability with parameters)
There are finitely many $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the $\Delta^0_2$ degrees.

1. The automorphism group of $\mathcal{D}_T(\leq 0')$ is countable.
Applications

Theorem (Biinterpretability with parameters)

There are finitely many $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the $\Delta^0_2$ degrees.

1. The automorphism group of $\mathcal{D}_T(\leq 0')$ is countable.
2. Every automorphism $\pi$ of $\mathcal{D}_T(\leq 0')$ has an arithmetic presentation.
Applications

Theorem (Biinterpretability with parameters)

There are finitely many $\Delta_2^0$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the $\Delta_2^0$ degrees.

1. The automorphism group of $\mathcal{D}_T(\leq 0')$ is countable.
2. Every automorphism $\pi$ of $\mathcal{D}_T(\leq 0')$ has an arithmetic presentation.
3. Every relation $\mathcal{R} \subseteq \mathcal{D}_T(\leq 0')$ induced by an arithmetically definable degree invariant relation is definable with finitely many $\Delta_2^0$ parameters. If $\mathcal{R}$ is invariant under automorphisms then it is definable.
Applications

Theorem (Biinterpretability with parameters)

There are finitely many $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the $\Delta^0_2$ degrees.

1. The automorphism group of $\mathcal{D}_T(\leq 0')$ is countable.
2. Every automorphism $\pi$ of $\mathcal{D}_T(\leq 0')$ has an arithmetic presentation.
3. Every relation $\mathcal{R} \subseteq \mathcal{D}_T(\leq 0')$ induced by an arithmetically definable degree invariant relation is definable with finitely many $\Delta^0_2$ parameters. If $\mathcal{R}$ is invariant under automorphisms then it is definable.
4. $\mathcal{D}_T(\leq 0')$ is rigid if and only if $\mathcal{D}_T(\leq 0')$ is biinterpretable with first order arithmetic.
Part II: The structure of the enumeration degrees

Definition

\( A \leq_{e} B \) if there is a c.e. set \( W \), such that

\[
A = W(B) = \{ x \mid \exists D (\langle x, D \rangle \in W \land D \subseteq B) \}.
\]
Part II: The structure of the enumeration degrees

Definition

\( A \leq_e B \) if there is a c.e. set \( W \), such that

\[
A = W(B) = \{ x \mid \exists D (\langle x, D \rangle \in W \land D \subseteq B) \}.
\]

- \( A \equiv_e B \) if \( A \leq_e B \) and \( B \leq_e A \).
Part II: The structure of the enumeration degrees

**Definition**

A $\leq_e B$ if there is a c.e. set $W$, such that

$$A = W(B) = \{x \mid \exists D (\langle x, D \rangle \in W \land D \subseteq B)\}.$$ 

- A $\equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.
- The enumeration degree of a set $A$ is $d_e(A) = \{B \mid A \equiv_e B\}$. 
Part II: The structure of the enumeration degrees

Definition

\( A \leq_e B \) if there is a c.e. set \( W \), such that

\[
A = W(B) = \{ x \mid \exists D ( \langle x, D \rangle \in W \& D \subseteq B ) \}.
\]

- \( A \equiv_e B \) if \( A \leq_e B \) and \( B \leq_e A \).
- The enumeration degree of a set \( A \) is \( d_e(A) = \{ B \mid A \equiv_e B \} \).
- \( d_e(A) \leq d_e(B) \) iff \( A \leq_e B \).
Part II: The structure of the enumeration degrees

Definition

\( A \leq_e B \) if there is a c.e. set \( W \), such that

\[
A = W(B) = \{ x \mid \exists D (\langle x, D \rangle \in W \& D \subseteq B) \}.
\]

- \( A \equiv_e B \) if \( A \leq_e B \) and \( B \leq_e A \).
- The enumeration degree of a set \( A \) is \( d_e(A) = \{ B \mid A \equiv_e B \} \).
- \( d_e(A) \leq d_e(B) \) iff \( A \leq_e B \).
- The least element: \( 0_e = d_e(\emptyset) \), the set of all c.e. sets.
**Part II: The structure of the enumeration degrees**

**Definition**

\( A \leq_e B \) if there is a c.e. set \( W \), such that

\[
A = W(B) = \{ x \mid \exists D (\langle x, D \rangle \in W \& D \subseteq B) \}.
\]

- \( A \equiv_e B \) if \( A \leq_e B \) and \( B \leq_e A \).
- The enumeration degree of a set \( A \) is \( d_e(A) = \{ B \mid A \equiv_e B \} \).
- \( d_e(A) \leq d_e(B) \) iff \( A \leq_e B \).
- The least element: \( 0_e = d_e(\emptyset) \), the set of all c.e. sets.
- The least upper bound: \( d_e(A) \lor d_e(B) = d_e(A \oplus B) \).
Part II: The structure of the enumeration degrees

**Definition**

*A ≤² B* if there is a c.e. set *W*, such that

\[ A = W(B) = \{ x | \exists D(\langle x, D \rangle \in W \& D \subseteq B) \}. \]

- *A ≡² B* if *A ≤² B* and *B ≤² A*.
- The enumeration degree of a set *A* is *d_e(A) = \{ B | A ≡² B \}.*
- *d_e(A) ≤² d_e(B)* iff *A ≤² B*.
- The least element: *0_e = d_e(∅)*, the set of all c.e. sets.
- The least upper bound: *d_e(A) ∨ d_e(B) = d_e(A ⊕ B).*
- The enumeration jump: *d_e(A)′ = d_e(K_A ⊕ \overline{K_A})*, where *K_A = \{ \langle e, x \rangle | x \in W_e(A) \}.*
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

\[ A \leq_T B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}. \]
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$$A \leq_T B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$ 

A set $A$ is *total* if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is *total* if it contains a total set. The set of total degrees is denoted by $\text{TOT}$. 

Mariya I. Soskova (Sofia University)
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

**Proposition**

$$A \leq_T B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$ 

A set $A$ is *total* if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is *total* if it contains a total set. The set of total degrees is denoted by $\mathcal{TOT}$.

The embedding $\iota : \mathcal{D}_T \rightarrow \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

**Proposition**

$$A \leq_T B \iff A \oplus \bar{A} \leq_e B \oplus \bar{B}.$$ 

A set $A$ is *total* if $A \equiv_e A \oplus \bar{A}$. An enumeration degree is *total* if it contains a total set. The set of total degrees is denoted by $\mathcal{TOT}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \bar{A})$, preserves the order, the least upper bound and the jump operation.

$$(\mathcal{D}_T, \leq_T, \lor', \langle 0_T \rangle) \cong (\mathcal{TOT}, \leq_e, \lor', \langle 0_e \rangle) \subseteq (\mathcal{D}_e, \leq_e, \lor', \langle 0_e \rangle)$$
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

\[ A \leq_T B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}. \]

A set $A$ is total if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is total if it contains a total set. The set of total degrees is denoted by $\mathcal{TOT}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

\[ (\mathcal{D}_T, \leq_T, \lor', \bot_T) \cong (\mathcal{TOT}, \leq_e, \lor', \bot_e) \leq (\mathcal{D}_e, \leq_e, \lor', \bot_e) \]

If $x \in \mathcal{D}_T$ then we will call $\iota(x)$ the image of $x$. 
Theorem (Kalimullin)

The enumeration jump is first order definable in $D_e$. 

Theorem (Cai, Ganchev, Lempp, Miller, S)

The set of total enumeration degrees is first order definable in the enumeration degrees.
Theorem (Kalimullin)

The enumeration jump is first order definable in $D_e$.

Theorem (Cai, Ganchev, Lempp, Miller, S)

The set of total enumeration degrees is first order definable in the enumeration degrees.
Definability in the enumeration degrees

Theorem (Kalimullin)

The enumeration jump is first order definable in $D_e$.

Theorem (Cai, Ganchev, Lempp, Miller, S)

The set of total enumeration degrees is first order definable in the enumeration degrees.

Definition

A Turing degree $a$ is c.e. in a Turing degree $x$ if some $A \in a$ is c.e. in some $X \in x$. 
Theorem (Kalimullin)

The enumeration jump is first order definable in $\mathcal{D}_e$.

Theorem (Cai, Ganchev, Lempp, Miller, S)

The set of total enumeration degrees is first order definable in the enumeration degrees.

Definition

A Turing degree $a$ is c.e. in a Turing degree $x$ if some $A \in a$ is c.e. in some $X \in x$.

Theorem (Cai, Ganchev, Lempp, Miller, S)

The image of the relation “c.e. in” in the enumeration degrees is first order definable in $\mathcal{D}_e$. 
The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
\[ \{ x \in TOT \mid d_e(A) \leq x \} \supseteq \{ x \in TOT \mid d_e(B) \leq x \}. \]
The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
\[ \{ x \in TOT \mid d_e(A) \leq x \} \supseteq \{ x \in TOT \mid d_e(B) \leq x \} \].

Corollary

The total enumeration degrees form a definable automorphism basis of the enumeration degrees.
The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
\[ \{ x \in \mathcal{TOT} \mid d_e(A) \leq x \} \supseteq \{ x \in \mathcal{TOT} \mid d_e(B) \leq x \}. \]

Corollary

The total enumeration degrees form a definable automorphism basis of the enumeration degrees.

- If \( \mathcal{D}_T \) is rigid then \( \mathcal{D}_e \) is rigid.
The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
\[ \{ x \in \mathcal{T} \mathcal{O} \mathcal{T} \mid d_e(A) \leq x \} \supseteq \{ x \in \mathcal{T} \mathcal{O} \mathcal{T} \mid d_e(B) \leq x \}. \]

Corollary

The total enumeration degrees form a definable automorphism basis of the enumeration degrees.

- If \( \mathcal{D}_T \) is rigid then \( \mathcal{D}_e \) is rigid.
- The automorphism analysis for the enumeration degrees follows.
The total degrees as an automorphism base

**Theorem (Selman)**

A is enumeration reducible to B if and only if
\[ \{ x \in TOT \mid d_e(A) \leq x \} \supseteq \{ x \in TOT \mid d_e(B) \leq x \} . \]

**Corollary**

The total enumeration degrees form a definable automorphism basis of the enumeration degrees.

- If \( D_T \) is rigid then \( D_e \) is rigid.
- The automorphism analysis for the enumeration degrees follows.
- The total degrees below \( 0_e^{(5)} \) are an automorphism base of \( D_e \).
Towards a better automorphism base of $D_e$

**Theorem**

There are total $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the total $\Delta^0_2$ enumeration degrees.
Towards a better automorphism base of $\mathcal{D}_e$

**Theorem**

There are total $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the total $\Delta^0_2$ enumeration degrees.

1. The parameters $\bar{\rho}$ code an indexing of the image of the c.e. Turing degrees.

2. The parameters $\bar{\rho}$ code an indexing of the image of a uniformly low set $Z$.

3. Every low total $\Delta^0_2$ enumeration degree is uniquely positioned with respect to the image of the c.e. degrees and the image of $Z$.

4. Every total $\Delta^0_2$ enumeration degree is uniquely positioned with respect to the low total $\Delta^0_2$ enumeration degrees.
Towards a better automorphism base of $\mathcal{D}_e$

**Theorem**

There are total $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the total $\Delta^0_2$ enumeration degrees.

1. The parameters $\vec{p}$ code an indexing of the image of the c.e. Turing degrees.
2. The parameters $\vec{p}$ code an indexing of the image of a uniformly low set $\mathcal{Z}$. 
Towards a better automorphism base of $D_e$

**Theorem**

There are total $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the total $\Delta^0_2$ enumeration degrees.

1. The parameters $\bar{\rho}$ code an indexing of the image of the c.e. Turing degrees.
2. The parameters $\bar{\rho}$ code an indexing of the image of a uniformly low set $\mathcal{Z}$.
3. Every low total $\Delta^0_2$ enumeration degree is uniquely positioned with respect to the image of the c.e. degrees and the image of $\mathcal{Z}$.
Towards a better automorphism base of $\mathcal{D}_e$

**Theorem**

There are total $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the total $\Delta^0_2$ enumeration degrees.

1. The parameters $\vec{\rho}$ code an indexing of the image of the c.e. Turing degrees.
2. The parameters $\vec{\rho}$ code an indexing of the image of a uniformly low set $\mathcal{Z}$.
3. Every low total $\Delta^0_2$ enumeration degree is uniquely positioned with respect to the image of the c.e. degrees and the image of $\mathcal{Z}$.
4. Every total $\Delta^0_2$ enumeration degree is uniquely positioned with respect to the low total $\Delta^0_2$ enumeration degrees.
Theorem

1. Every low $\Delta^0_2$ enumeration degree is uniquely positioned with respect to the image of the c.e. Turing degrees and the low 3-c.e. enumeration degrees.
An improvement

Theorem

1. Every low $\Delta^0_2$ enumeration degree is uniquely positioned with respect to the image of the c.e. Turing degrees and the low 3-c.e. enumeration degrees.

2. Every low 3-c.e. enumeration degree is uniquely positioned with respect to the image of the c.e. Turing degrees.
An improvement

Theorem

1. Every low $\Delta^0_2$ enumeration degree is uniquely positioned with respect to the image of the c.e. Turing degrees and the low 3-c.e. enumeration degrees.

2. Every low 3-c.e. enumeration degree is uniquely positioned with respect to the image of the c.e. Turing degrees.

If $\vec{p}$ defines a model of arithmetic $M$ and an indexing of the images of the c.e. Turing degrees then $\vec{p}$ defines an indexing of the total $\Delta^0_2$ enumeration degrees.
Stepping outside the local structure

New Goal

Using parameters $\vec{p}$ that index the image of the c.e. degrees define an indexing of the images of all Turing degrees that are c.e. in and above some $\Delta^0_2$ Turing degree.
Stepping outside the local structure

New Goal

Using parameters $\vec{p}$ that index the image of the c.e. degrees define an indexing of the images of all Turing degrees that are c.e. in and above some $\Delta^0_2$ Turing degree.

$$\psi(e_0^M, e_1^M) = \iota(d_T(Y)), \text{ where } Y = W_{e_0}^X \text{ and } X = \{e_1\}^{\emptyset'}. $$
Stepping outside the local structure

New Goal

Using parameters $\vec{p}$ that index the image of the c.e. degrees define an indexing of the images of all Turing degrees that are c.e. in and above some $\Delta^0_2$ Turing degree.

$$\psi(e_0^M, e_1^M) = \iota(d_T(Y)),$$ where $Y = W_{e_0}^X$ and $X = \{e_1\}^{0'}$.

If we succeed then relativizing the previous step to any total $\Delta^0_2$ enumeration degree we can extend this to an indexing of the image of $\bigcup_{x \leq_{T0'} x'} [x, x']$. 
Stepping outside the local structure

New Goal

Using parameters $\vec{p}$ that index the image of the c.e. degrees define an indexing of the images of all Turing degrees that are c.e. in and above some $\Delta^0_2$ Turing degree.

$$\psi(e^M_0, e^M_1) = \iota(d_T(Y)), \text{ where } Y = W^X_{e_0} \text{ and } X = \{ e_1 \}^{\emptyset'}.$$

- If we succeed then relativizing the previous step to any total $\Delta^0_2$ enumeration degree we can extend this to an indexing of the image of $\bigcup_{x \leq_T 0'} [x, x'].$
- We will use that the image of the relation ‘c.e. in’ and the enumeration jump are definable.
C.e. in and above a $\Delta^0_2$ degree

Suppose that $x$ is $\Delta^0_2$ and $y$ is c.e. in and above $x$.

1. If $y \geq 0'$ then we use Shoenfield's jump inversion theorem to find a $\Delta^0_2$ degree $z$ such that $z' = y$.
C.e. in and above a $\Delta^0_2$ degree

Suppose that $x$ is $\Delta^0_2$ and $y$ is c.e. in and above $x$.

1. If $y \geq 0'$ then we use Shoenfield’s jump inversion theorem to find a $\Delta^0_2$ degree $z$ such that $z' = y$.

2. Otherwise using Sacks’ splitting theorem we can represent $y$ as $a_1 \lor a_2$, where $a_1$ and $a_2$ are low and c.e.a. relative to $x$ which avoid the cone above $0'$. 

Mariya I. Soskova (Sofia University)
Suppose that $x$ is $\Delta^0_2$ and $y$ is c.e. in and above $x$.

1. If $y \geq 0'$ then we use Shoenfield’s jump inversion theorem to find a $\Delta^0_2$ degree $z$ such that $z' = y$.

2. Otherwise using Sacks’ splitting theorem we can represent $y$ as $a_1 \lor a_2$, where $a_1$ and $a_2$ are low and c.e.a. relative to $x$ which avoid the cone above $0'$.

3. Define an indexing of all low and c.e.a. relative to $x$ such avoid the cone above $0'$. 
C.e. in and above a $\Delta^0_2$ degree

Suppose that $x$ is $\Delta^0_2$ and $y$ is c.e. in and above $x$.

1. If $y \geq 0'$ then we use Shoenfield’s jump inversion theorem to find a $\Delta^0_2$ degree $z$ such that $z' = y$.

2. Otherwise using Sacks’ splitting theorem we can represent $y$ as $a_1 \vee a_2$, where $a_1$ and $a_2$ are low and c.e.a. relative to $x$ which avoid the cone above $0'$.

3. Define an indexing of all low and c.e.a. relative to $x$ such avoid the cone above $0'$.

We can define the set of images of low relative to $x$ degrees that are c.e. in and above $x$ and avoid the cone above $0'$. 
C.e. in and above a $\Delta^0_2$ degree: complicated case

**Theorem**

If $Y$ and $W$ are c.e. sets and $A$ is a low c.e. set such that $W \not\leq_T A$ and $Y \not\leq_T A$ then there are sets $U$ and $V$ computable from $W$ such that:

1. $V \leq_T Y \oplus U$
2. $V \not\leq_T A \oplus U$

Relative to $X$ and with $W = \emptyset'$ we get:

Within the class of low and c.e. degrees relative to $X$ which do not compute $\emptyset'$, $y$ is uniquely positioned with respect to the $\Delta^0_2$ Turing degrees.
C.e. in and above a $\Delta^0_2$ degree: complicated case

**Theorem**

If $Y$ and $W$ are c.e. sets and $A$ is a low c.e. set such that $W \not\leq_T A$ and $Y \not\leq_T A$ then there are sets $U$ and $V$ computable from $W$ such that:

1. $V \leq_T Y \oplus U$
2. $V \not\leq_T A \oplus U$

Relative to $X$ and with $W = \emptyset'$ we get:
C.e. in and above a $\Delta^0_2$ degree: complicated case

**Theorem**

If $Y$ and $W$ are c.e. sets and $A$ is a low c.e. set such that $W \nleq_T A$ and $Y \nleq_T A$ then there are sets $U$ and $V$ computable from $W$ such that:

1. $V \leq_T Y \oplus U$
2. $V \nleq_T A \oplus U$

Relative to $X$ and with $W = \emptyset'$ we get:

Within the class of low and c.e.a degrees relative to $x$ which do not compute $\emptyset'$, $y$ is uniquely positioned with respect to the $\Delta^0_2$ Turing degrees.
The rest of the total enumeration degrees

**Theorem**

Let $\vec{p}$ are parameters that index the image of the c.e. Turing degrees then $\vec{p}$ index $\bigcup_{x \leq_T 0'} [x, x']$.

Next Goal

Extend to an indexing of the image of all $\Delta^0_3$ Turing degrees.

**Theorem**

There are high $\Delta^0_2$ degrees $h_1$ and $h_2$ such that every $\Delta^0_3$ Turing degree $g$ satisfies $(h_1 \lor g) \land (h_2 \lor g) = g$.

Note that $h_i \lor g \in [h_i, h_i']$ thus we have a way to identify this degree and hence we have a way to identify $g$. 

Mariya I. Soskova ( Sofia University )
The rest of the total enumeration degrees

**Theorem**

Let $\vec{p}$ are parameters that index the image of the c.e. Turing degrees then $\vec{p}$ index $\bigcup_{x \leq T^0_0} [x, x']$.

**Next Goal**

Extend to an indexing of the image of all $\Delta^0_3$ Turing degrees.
The rest of the total enumeration degrees

**Theorem**

Let \( \vec{p} \) are parameters that index the image of the c.e. Turing degrees then \( \vec{p} \) index \( \bigcup_{x \leq_{T} 0'} [x, x'] \).

**Next Goal**

Extend to an indexing of the image of all \( \Delta^0_3 \) Turing degrees.

**Theorem**

There are high \( \Delta^0_2 \) degrees \( h_1 \) and \( h_2 \) such that every 2-generic \( \Delta^0_3 \) Turing degree \( g \) satisfies \( (h_1 \lor g) \land (h_2 \lor g) = g \).
The rest of the total enumeration degrees

**Theorem**

Let $\vec{p}$ are parameters that index the image of the c.e. Turing degrees then $\vec{p}$ index $\bigcup_{x \leq_T 0'} [x, x']$.

**Next Goal**

Extend to an indexing of the image of all $\Delta^0_3$ Turing degrees.

**Theorem**

There are high $\Delta^0_2$ degrees $h_1$ and $h_2$ such that every 2-generic $\Delta^0_3$ Turing degree $g$ satisfies $(h_1 \vee g) \land (h_2 \vee g) = g$.

Note that $h_i \vee g \in [h_i, h_i']$ thus we have a way to identify this degree and hence we have a way to identify $g$. 
And now we iterate!

Theorem

Let $n$ be a natural number and $\tilde{p}$ be parameters that index the image of the c.e. Turing degrees. There is a definable from $\tilde{p}$ indexing of the total $\Delta^0_{n+1}$ sets.
Consequences

1. There is a finite automorphism base for the enumeration degrees consisting of total $\Delta^0_2$ enumeration degrees:
Consequences

1. There is a finite automorphism base for the enumeration degrees consisting of total $\Delta_2^0$ enumeration degrees:

2. The image of the c.e. Turing degrees is an automorphism base for $\mathcal{D}_e$. 
Consequences

1. There is a finite automorphism base for the enumeration degrees consisting of total $\Delta^0_2$ enumeration degrees:

2. The image of the c.e. Turing degrees is an automorphism base for $D_e$.

3. If the structure of the c.e. Turing degrees is rigid then so is the structure of the enumeration degrees.
Consequences

1. There is a finite automorphism base for the enumeration degrees consisting of total $\Delta^0_2$ enumeration degrees:

2. The image of the c.e. Turing degrees is an automorphism base for $D_e$.

3. If the structure of the c.e. Turing degrees is rigid then so is the structure of the enumeration degrees.

Question

1. *Can every automorphism of the Turing degrees be extended to an automorphism of the enumeration degrees?*
Consequences

1. There is a finite automorphism base for the enumeration degrees consisting of total $\Delta^0_2$ enumeration degrees:
2. The image of the c.e. Turing degrees is an automorphism base for $\mathcal{D}_e$.
3. If the structure of the c.e. Turing degrees is rigid then so is the structure of the enumeration degrees.

Question

1. Can every automorphism of the Turing degrees be extended to an automorphism of the enumeration degrees?
2. Can we extend automorphisms of the c.e. degrees to automorphisms of $\mathcal{D}_T$ or of $\mathcal{D}_e$?