The complexity of Π^1_1 randomness

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Background

Martin-Löf suggested (1970) studying Δ^1_1 randomness. A real is Δ^1_1 random if it avoids all hyperarithmetic null sets; equivalently if it is ML random relative to $\mathbf{0}^{(\alpha)}$ for all computable ordinals α.

In general, measure theory in the context of effective descriptive set theory was studied by Spector, Sacks, Tanaka, Kechris, Stern, ...

More recently, Hjorth and Nies studied how notions of algorithmic randomness behave when c.e. is replaced by Π^1_1. For example they define the notion of Π^1_1-ML-randomness.
This study was continued by Chong, Nies and Yu.

A question of complexity

How complicated is it to compute ω_1^{ck}?

Terminology

A real x preserves ω_1^{ck} if $\omega_1^x = \omega_1^{\mathrm{ck}}$. Otherwise it collapses ω_1^{ck}.

The set of reals which collapse ω_1^{ck} is Π^1_1 but not Σ^1_1. It is Borel, since it is Σ^1_1 relative to Kleene's O (the complete Π^1_1 subset of ω).

Steel stated that the set of reals which preserve ω_1^{ck} is $\boldsymbol{\Pi}^0_{\omega_1^{\mathrm{ck}}+2}$, but not simpler.

Test case: Cohen generics

However, the situation is different if we restrict ourselves to Cohen generic reals.

Theorem

The following are equivalent for a Δ^1_1 Cohen generic real g:

1. *g meets all dense Σ^1_1 sets of strings;*
2. *g meets or avoids all Σ^1_1 sets of strings;*
3. *g preserves ω_1^{ck}.*

Corollary

The set of Δ^1_1 Cohen generic reals which preserve ω_1^{ck} is $\mathbf{\Pi^0_2}$.

(Earlier, Slaman and Greenberg noticed that if g is Δ^1_1 generic and preserves ω_1^{ck} then it meets or avoids all Π^1_1 sets of strings. The latter however is a weaker condition.)

Π^1_1 randomness

Fact (Spector,Sacks)

Almost every real preserves ω_1^{ck}.

Fact (Kechris)

There is a largest null Π^1_1 set.

A real avoiding this null set is called Π^1_1 random.

Theorem (Stern;Chong,Nies,Yu)

A Δ^1_1 random real preserves ω_1^{ck} if and only if it is Π^1_1 random.

Question (Yu)

What is the complexity of the set of Π^1_1 random reals?

Higher notions of computability and randomness

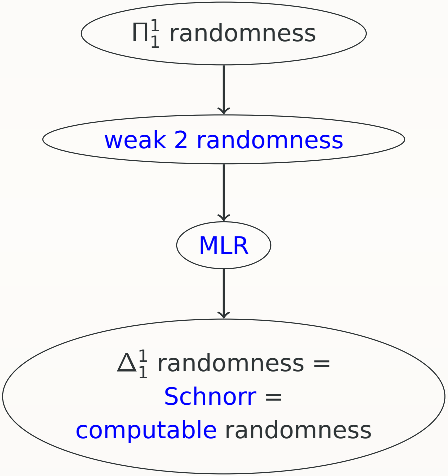
Much of the development of higher randomness relies on the analogy between Π^1_1 and Σ^0_1:

- For subsets of ω, Π^1_1 is the same as Σ^0_1 over $L_{\omega_1^{\mathrm{ck}}}$;
- For subsets of 2^ω, Π^1_1 is the same as Σ^0_1 over $L_{\omega_1^x}[x]$, uniformly.

We use the colour blue to denote concepts in which the innermost existential quantifier has been changed to range over ω_1^{ck}. For example,

- $\Sigma^0_1 = \Pi^1_1$ (for subsets of ω and for open subsets of 2^ω);
- Π^0_2 is the intersection $\bigcap_{n<\omega} U_n$ of a uniform sequence of Σ^0_1 sets.
- A real is MLR if it avoids all effectively null Π^0_2 sets. Also denoted by Π^1_1-MLR.
- A real is weakly 2 random if it avoids all null Π^0_2 sets. Also denoted by Π^1_1-weakly 2 random.

Quick summary: higher randomness notions



omitted: difference randomness, ...

Short Π^1_1 sets

Every $\mathbf{\Pi^1_1}$ subset of 2^ω is the union $\bigcup_{\alpha<\omega_1} A_\alpha$ with each A_α Borel. If it is Π^1_1 then each A_α is $\Delta^1_1(\alpha)$ (uniformly in any code for α).

Definition

A Π^1_1 set $A \subseteq 2^\omega$ is short if it is the uniform union $\bigcup_{\alpha<\omega_1^{\mathrm{ck}}} A_\alpha$ of Δ^1_1 sets.

Using the fact that Δ^1_1 sets can be approximated from above (in the sense of measure) by open Δ^1_1 sets:

Lemma

A short Π^1_1 set A can be approximated from above by Π^1_1 open sets: for all ϵ there is a Π^1_1 open (Σ^0_1) set $U^\epsilon \supseteq A$ such that $\lambda(U^\epsilon - A) < \epsilon$.

In fact we can arrange that $\lambda\left(U^\epsilon_\alpha - A_\alpha\right) < \epsilon$ for all α (and anyway it happens on a closed and unbounded set). Finding U^ϵ is uniform in A and ϵ.

Approximating with Π^0_2 sets

Let $B = \bigcap A_n$ be a uniform intersection of short Π^1_1 sets.

- For $\alpha < \omega_1^{\mathrm{ck}}$ we let $B_\alpha = \bigcap_n A_{n,\alpha}$.
- We let $B_{<\omega_1^{\mathrm{ck}}} = \bigcup_{\alpha<\omega_1^{\mathrm{ck}}} B_\alpha$.

Note that if $x \in B - B_{<\omega_1^{\mathrm{ck}}}$ then x collapses ω_1^{ck}.

Proposition

Suppose that x is Δ^1_1 random and collapses ω_1^{ck}. Then there is a Π^0_2 set G such that $x \in G - G_{<\omega_1^{\mathrm{ck}}}$.

Proof.

Let L be a computable operator taking reals to linear orderings such that $L^x \cong \omega_1^{\mathrm{ck}}$. For $n < \omega$ let

$$A_n = \left\{y \,:\, \mathrm{otp}(L^y \restriction_n) < \omega_1^{\mathrm{ck}}\right\}$$

and let $B = \bigcap_n A_n$. Then $x \in B - B_{<\omega_1^{\mathrm{ck}}}$.

Approximate each A_n by U_n^ϵ; let $G = \bigcap_{n,\epsilon} U_{n,\epsilon}$. For all α, $G_\alpha - A_\alpha$ is null (and Δ^1_1), so $x \in G - G_{<\omega_1^{\mathrm{ck}}}$. □

The Borel rank

Lemma

Let G be Π^0_2 and let $P \subseteq G$ be Π^0_1 (a closed Σ^1_1 set). Then $P \subseteq G_{<\omega_1^{ck}}$.

Proof.

Say $G = \bigcap_n U_n$. By compactness, for all n there is some $\alpha < \omega_1^{ck}$ such that $P_\alpha \subseteq U_{n,\alpha}$. By admissibility, these are all bounded below ω_1^{ck}. □

For any set G, let G^* be the union of all Π^0_1 subsets of G.

Lemma

If G is Π^1_1 then $G - G^*$ is null.

If G is $\boldsymbol{\Pi^0_2}$ then $G - G^*$ is also $\boldsymbol{\Pi^0_2}$.

Corollary

The set of Π^1_1 random reals is $\boldsymbol{\Pi^0_3}$.

Techniques of Yu Liang's show that it is not $\boldsymbol{\Sigma^0_3}$.

Forcing with Π^0_1 sets of positive measure

Proposition

If x is sufficiently generic for forcing with Π^0_1 classes of positive measure then x is Π^1_1 random.

Proof.

Let P be effectively closed of positive measure.
Let $H = \bigcap_n U_n$ be Π^0_2.
If P is not almost contained in H then for some n, $P - U_n$ is not null, extends P and forces that $x \notin H$.
Otherwise, P is almost contained in H^*, so we can find $P' \subseteq H^*$ such that $\lambda(P \cap P') > 0$. □

Lowness for Π^1_1 randomness

Theorem (Hjorth,Nies)

If $a \in 2^\omega$ is not hyperarithmetic then a is not low for Π^1_1-MLR.

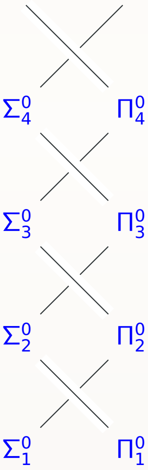
Let $a \notin \Delta^1_1$. There is some $\Pi^1_1(a)$ and open U of measure < 1 which cannot be covered by a Π^1_1 open set of measure < 1. In other words, U intersects every Π^0_1 set, in fact has positive intersection with each such set. By induction, U^n has the same property. If x is sufficiently generic for forcing with Π^0_1 sets of positive measure then $x \in U^n$ for all n, and so $x \notin \Pi^1_1(a)$-MLR.

Corollary

A real is low for Π^1_1 randomness if and only if it is hyperarithmetic.

A refinement of the question

The parameter for the $\mathbf{\Pi^0_3}$ is complicated. We effectivise the complexity question by considering the higher arithmetic hierarchy.



The effective Borel rank: a lower bound

If G is Π^0_2 then $G - G^*$ is the intersection of Π^1_1 open sets. But not uniformly so: $P \subset G$ is a c.e. event but not decidable.

Theorem

The set of Π^1_1 randoms is not Π^0_3.

Proposition

If a Π^0_3 set is co-null then either it contains a hyperarithmetic real or a real which collapses ω_1^{ck}.

Finite change approxmations

The higher limit lemma says that x is computable from Kleene's O (the complete Π^1_1 subset of ω) if and only if $x = \lim_{s<\omega_1^{ck}} x_s$ with $\langle x_s \rangle$ uniformly hyperarithmetic. The limit means that for all $n < \omega$ there is some $s < \omega_1^{ck}$ such that $x_t \upharpoonright_n = x \upharpoonright_n$ for all $t \in [s, \omega_1^{ck})$.

A stronger property is having a finite change approximation: for all n, $\langle x_s \upharpoonright_n \rangle$ changes only finitely often.

Lemma

If x has a finite-change approximation then either x is hyperarithmetic or it collapses ω_1^{ck}.

Proof.

We may assume that for all $s < \omega_1^{ck}$, $x_s = \lim_{t<s} x_t$. If $x \neq x_s$ for all s then the function taking x to the least s such that $x_s \upharpoonright_n = x \upharpoonright_n$ is unbounded in ω_1^{ck}. □

Proposition

Every co-null Π^0_3 set contains a real which has a finite-change approximation.

Proof.

Let $F = \bigcap_n F_n$ be a co-null Π^0_3 set. So each F_n is co-null. Each F_n is the union of an increasing sequence $\langle F_{n,m} \rangle_{m<\omega}$ of Π^0_1 sets; so $\lim_m \lambda(F_{n,m}) = 1$.
Idea: let m_0 be the least such that $\lambda(F_{0,m_0}) \geqslant 1/2$. Let $x(0) \in \{0,1\}$ such that $\lambda(F_{0,m_0} | x(0)) \geqslant 1/2$.
Next, let m_1 be least such that $\lambda(F_{0,m_0} \cap F_{1,m_1} | x(0)) \geqslant 1/4$. Let $x(1)$ be such that $\lambda(F_{0,m_0} \cap F_{1,m_1} | x(0)x(1)) \geqslant 1/4$. And so on.

Our guess for what m_0 is changes at most m_0 many times, and so our guess for $x(0)$ changes at most $2m_0$ many times.
Within any interval of stages at which our guess for $x(0)$ and $m(0)$ is constant, our guess for what m_1 is changes finitely many times (perhaps more than the final m_1). And so on. Note: it is not enough to check only the final interval (the correct m_0 and $x(0)$ guess). □

The effective Borel rank: an upper bound

Theorem
The set of Π^1_1 randoms is Π^0_5.

To show this, for any Π^0_2 set G we show that $G - G_{\omega_1^{ck}}$ is a Π^0_4 set (uniformly in G).

*** I am lying. Try to catch me ***

For $x \in G$ let η^x be the least α such that $x \in G_\alpha$. So we want to capture those x for which $\eta^x = \omega_1^{ck}$.

The problem is that the intersection $\bigcap_\alpha \{x : \eta^x > \alpha\}$ ranges over computable ordinals, not natural numbers.

The effective Borel rank: an upper bound

Instead we need to consider all computable linear orderings, not only the well-founded ones. For $e < \omega$ let A_e be the set of x such that the well-founded part of L_e is smaller than η^x. This is Σ^0_1. If we take the intersection of all A_e we get nothing, since for some e, the well-founded part of L_e is ω_1^{ck}.

To take care of these, let B_e be the set of x such that η^x embeds in some proper initial segment of L_e. This is Σ^0_3. If L_e is a Harrison linear ordering then $B_e = 2^\omega$. So $G - G_{<\omega_1^{\mathrm{ck}}} \subseteq \bigcap_e (A_e \cup B_e)$.
On the other hand if $\eta^x < \omega_1^{\mathrm{ck}}$ and $L_e \cong \eta^x$ then $x \notin A_e \cup B_e$. Hence

$$\bigcap_e (A_e \cup B_e) = G - G_{\omega_1^{\mathrm{ck}}}.$$

The effective Borel rank

So the set of Π^1_1 randoms is Π^0_5 and not Π^0_3. The only unknown left is: is it Σ^0_4?

Proposition

The set of Π^1_1 randoms is not Σ^0_4 if and only if every Π^0_3 set of positive measure contains a real which collapses ω_1^{ck}.

Proof.

In the interesting direction: suppose that A is Π^0_3, not null, and contains no reals which collapse ω_1^{ck}. We may assume that every $x \in A$ is Π^1_1-MLR, so every $x \in A$ is Π^1_1-random. Let $B = \bigcup_{\sigma \in 2^{<\omega}} \sigma\hat{\ }B$. Then B is Σ^0_4 (and so is Σ^1_1) and every $x \in B$ is Π^1_1 random. By the Lebesgue density theorem, B is co-null. It is contained in the smallest co-null Σ^1_1 set, and so must equal it. $\square$

Attempting a separation between Π^1_1 randomness and weak 2 randomness

Suppose that x has a finite-change approximation $\langle x_s \rangle$. As we mentioned, we may assume that the set $\{x_s : s < \omega_1^{\mathrm{ck}}\} \cup \{x\}$ is closed. We say that x has a closed approximation (this is a weaker condition).

Proposition

If x has a closed approximation then it is not Π^1_1-weak 2 random.

Proof.

Let $U_n = \bigcup_{s<\omega_1^{\mathrm{ck}}} [x_s \upharpoonright n]$. Each U_n is clopen, and so $\bigcap_n U_n$ is the set $\{x_s : s < \omega_1^{\mathrm{ck}}\} \cup \{x\}$. This set is countable, and so is null. □

Corollary

The two halves of Ω are not Π^1_1-weakly 2 random, and so not Π^1_1 random.

So: if we want to separate Π^1_1 randomness from Π^1_1-weak 2 randomness, we cannot build a real with a closed approximation.

Closed and unbounded approximations

Lemma

Suppose that x is not hyperarithmetic, that $\langle x_s \rangle_{s<\omega_1^{ck}}$ is uniformly hyperarithmetic and that for all n, $\{s < \omega_1^{ck} : x_s \upharpoonright_n = x \upharpoonright_n\}$ is closed and unbounded. Then x collapses ω_1^{ck}.

We do not assume that $x = \lim_s x_s$ but we can adjust the approximation so that it is.

Proof.

Same proof. If the first occurrences of $x \upharpoonright_n$ are bounded below s then $x = x_s$. □

Proposition

There is a real x which is Π^1_1-weakly 2 random but has a club approximation.

The separation

Proposition

There is a real x which is Π^1_1-weakly 2 random but has a club approximation.

Proof sketch.

We approximate x, and for each e, if the e^{th} Σ^0_2 set $F_e = \bigcup_k F_{e,k}$ is co-null then we want $x \in F_e$. At some stage we are given $\sigma \prec x_s$ and a closed set H inherited from above such that $\lambda(H|\sigma) \geqslant \epsilon_e$. If F_e is co-null then we can find an extension $\tau \succ \sigma$ and some late enough k such that $\lambda(H \cap F_{e,k}|\tau) \geqslant \epsilon_e/2$ and we keep going; our guess for k (and τ) will change only finitely many times. However, if F_e is not co-null then we will go through all k first and only then discover that fact.

Idea: in this case discard σ. We have reserved in advance (as in Kučera coding) another σ' which we never touched before, also with $\lambda(H|\sigma') \geqslant \epsilon_e$. We now route the construction through σ'. We also made progress: we know that F_e is not co-null, so we can ignore it. □

Computing c.e. sets

Using Π^1_1 functionals we define a higher version of Turing reducibility. It is important that it is continuous (unlike relative hyperarithmetic reducibility).

The following theorem is an analogue of a result of Hirschfeldt and Miller characterising weak 2 randomness in terms of forming a minimal pair with $\mathbf{0}'$.

Theorem

The following are equivalent for a ML-random real x:

- *x is not Π^1_1 random.*
- *x computes a noncomputable c.e. set.*

Further questions

- Is the set of Π^1_1-weakly 2 random sets Σ^0_{2n} for any n?
- Can any nonhyperarithmetic set be joined above O with a Σ^1_1 generic? a Π^1_1 random?

Thank you