Infinite computations with random oracles

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Motivation

Many mathematical objects cannot be coded by integers, yet we can perform infinitary constructions with these objects

- constructing the algebraic closure of a field
- constructing the levels L_α of the constructible universe L

This motivates the study of infinitary computations, which give a precise meaning to various intuitive infinitary constructions.

Infinite time computations

Hamkins, Welch, Koepke and others studied Turing programs with infinite hardware and infinite time.

- analogies to Turing computability
- halting times
- relation with Π^1_1 and Σ^1_2 sets

Goals

- analogies to algorithmic randomness
- computability from a set of real oracles of positive measure

Infinite time Turing machines

Consider a Turing program which runs on the hardware of a Turing machine, but with infinite time (ITTM, Hamkins-Kidder 2000).

- the tape is a Turing tape
- the time is the ordinals (*Ord*)

The machine works as follows.

- the state is the *liminf* of the states at previous times
- the head moves to the *liminf* of its previous positions if this is finite, and to 0 otherwise
- the contents of each cell is the *liminf* of the contents at previous times

Example

Test if a symbol occurs infinitely often.
Test if a tree has an infinite branch.

Ordinal time/tape Turing machines

Consider a Turing program which runs on infinite hardware (OTM, Koepke 2006).

- the tape has length the ordinals
- the time is the ordinals

The transition in limit times is defined as follows.

- the state is the *liminf* of the states at previous times
- the head moves to the *liminf* of its previous positions
- the contents of each cell is the *liminf* of the contents at previous times

Example

Add ordinals. $\alpha + \beta$ is defined by

- $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
- $\alpha + \lambda = \sup_{\beta < \lambda} \alpha + \beta$ for limits λ

We represent α by a symbol in place α.

Computations

tape →

time ↓

	state	head	0	1	2	3	4	5	⋯	ω	⋯
0	0	0	0	0	0	0	0	0	⋯	0	⋯
1	1	1	1	0	0	0	0	0	⋯	0	⋯
2	0	2	1	0	0	0	0	0	⋯	0	⋯
3	1	3	1	0	1	0	0	0	⋯	0	⋯
4	0	4	1	0	1	0	0	0	⋯	0	⋯
5	1	5	1	0	1	0	0	0	⋯	0	⋯
6	0	6	1	0	1	0	1	0	⋯	0	⋯
7	1	7	1	0	1	0	1	0	⋯	0	⋯
8	0	8	1	0	1	0	1	0	⋯	0	⋯
9	1	9	1	0	1	0	1	0	⋯	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
ω	0	ω	1	0	1	0	1	0	⋯	0	⋯
$\omega+1$	1	$\omega+1$	1	0	1	0	1	0	⋯	1	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Computations from many oracles

Lemma (de Leuw-Moore-Shannon-Shapiro, Sacks)

If $A \subseteq 2^{\mathbb{N}}$, $\mu(A) > 0$, and $x \in 2^{\mathbb{N}}$ is computable from all $y \in A$, then x is computable.

Proof.

There is an interval U_s with $\frac{\mu(A \cap U_s)}{\mu(U_s)} > 0.5$ by the Lebesgue density theorem. Assume $\mu(A) > 0.5$.

Each bit is computed from some $s_0, ..., s_n$ with $\mu(U_{s_0} \cup ... \cup U_{s_n}) > 0.5$. □

Is this true for infinite computations?

- the machine can read all input bits during a computation
- we cannot list all possible input words

Computable sets

Definition

$x, y \in 2^{\mathbb{N}}$, $A \subseteq 2^{\mathbb{N}}$.

- x is *OTM-computable* from y ($x \leqslant_{OTM} y$) if there is an *OTM* P such that P halts on input y with output x ($P^y = x$).
- A is *OTM-computable* if there is an *OTM* P such that P halts on all inputs $x \in 2^{\mathbb{N}}$, and $x \in A$ iff $P^x = 0$.

- ITTMs compute Π^1_1 and Σ^1_1 sets of reals (Hamkins-Lewis)
- OTMs compute Δ^1_2 sets of reals (Koepke-Seyfferth)
- OTMs with ordinal oracles compute L (Koepke)

The constructible universe

The computable reals are those in some L_α for various machines.

Definition

$L_0 := \emptyset$

$L_{\alpha+1} := Def(L_\alpha, \in) := \{X \subseteq L_\alpha \mid X = \{x \in L_\alpha \mid (L_\alpha, \in) \models \varphi(x, a)\}$ for some $a \in L_\alpha$ and some first order formula $\varphi\}$

$L := \bigcup_{\alpha \in Ord} L_\alpha$

Halting times

Definition

Let η^x denote the supremum of halting times of *OTMs* with oracle x.

Lemma

The following conditions are equivalent for reals x, y.

- x *is* Δ^1_2 *in* y
- $x \leqslant_{OTM} y$
- $x \in L_{\eta^y}[y]$

OTM computations in L

Theorem

Suppose that $V = L$. There is a real x and a co-countable set $A \subseteq 2^{\mathbb{N}}$ such that

- *x is OTM-computable from every $y \in A$ but*
- *x is not OTM-computable.*

Corollary

Assume that $V = L$.

- *Let z denote the halting problem for OTMs. Then $z \leqslant_{OTM} x$ for every non-OTM-computable real x.*
- *For all reals x and y, $x \leqslant_{OTM} y$ or $y \leqslant_{OTM} x$.*

Cohen and random reals

Definition

Suppose that $x \in 2^{\mathbb{N}}$.

- x is Cohen over L_α if $x \in B$ for every comeager Borel set B with a Borel code in L_α.
- x is random over L_α if $x \in B$ for every measure 1 Borel set B with a Borel code in L_α.

- related to forcing in set theory
- related to randomness in computability

OTM computations from many oracles

Theorem

- *Suppose that for every $x \in 2^{\mathbb{N}}$, the set of random reals over $L[x]$ has measure 1 (iff every Σ^1_2 set is Lebesgue measurable).*

  If $A \subseteq 2^{\mathbb{N}}$ has positive measure and x is OTM-computable from every $y \in A$, then x is OTM-computable.

- *Suppose that for every $x \in 2^{\mathbb{N}}$, the set of Cohen reals over $L[x]$ is comeager (iff every Σ^1_2 set has the property of Baire).*

  If A is a nonmeager set with the property of Baire and x is OTM-computable from every $y \in A$, then x is OTM-computable.

OTM computations with ordinal parameters

Lemma

A real x is OTM-computable from y with ordinal oracles iff $x \in L[y]$, i.e. x is constructible from y.

In L, and in any model in which $(2^{\mathbb{N}})^L$ is not Lebesgue measurable, our question is trivial.

The following result follows easily from work of Judah-Shelah.

Theorem

There is a forcing $\mathbb{P}$ in L such that in any $\mathbb{P}$-generic extension of L there is a measure 1 set $A \subseteq 2^{\mathbb{N}}$ and

- *every $x \in A$ can be constructed from every $y \in A$*
- *A contains no constructible real*
- *$(2^{\mathbb{N}})^L$ has measure 0*

OTM computations with ordinal parameters

Theorem

- *Suppose that for every real x, there is a random real over $L[x]$.*

  If A has positive measure and $x \in 2^{\mathbb{N}}$ is constructible from each $y \in A$, then $x \in L$.

- *Suppose that for every real x, there is a Cohen real over $L[x]$.*

  If A is a nonmeager Borel set and $x \in 2^{\mathbb{N}}$ is constructible from each $y \in A$, then $x \in L$.

Question

Is it consistent that there is a nonconstructible real x and a Borel set A of measure 1 such that x is OTM-computable without parameters from every $y \in A$?

ITTM writable reals

Definition

Suppose that $x \in 2^{\mathbb{N}}$.

- Let λ^x denote the supremum of ITTM-writable ordinals (write-halt) with oracle x.
- Let ζ^x denote the supremum of ITTM-eventually writable ordinals (write-keep) with oracle x.
- Let Σ^x denote the supremum of ITTM-accidentally writable ordinals (write) with oracle x.

Theorem (Welch)

- *The reals writable (eventually writable, accidentally writable) in the oracle x are exactly those in $L_{\lambda^x}[x]$ ($L_{\zeta^x}[x]$, $L_{\Sigma^x}[x]$).*
- ζ^x, Σ^x *is the lexically least pair of ordinals with* $L_{\zeta^x}[x] \prec_{\Sigma_2} L_{\Sigma^x}[x]$
- λ^x *is minimal with* $L_{\lambda^x}[x] \prec_{\Sigma_1} L_{\zeta^x}[x]$.

ITTM computations from many oracles

Lemma

If x is Cohen generic over $L_{\Sigma+1}$ then

- $L_\lambda[x] \prec_{\Sigma_1} L_\zeta[x] \prec_{\Sigma_2} L_\Sigma[x]$.
- $\lambda^x = \lambda$, $\zeta^x = \zeta$ *and* $\Sigma^x = \Sigma$.

Theorem

Suppose that $A \subseteq 2^{\mathbb{N}}$ is a nonmeager Borel set and $x \in 2^{\mathbb{N}}$.

If x is writable (eventually writable, accidentally writable) in every oracle $y \in A$, then x is writable (eventually writable, accidentally writable).

Conjecture

Suppose that $A \subseteq 2^{\mathbb{N}}$ is a set with positive measure and $x \in 2^{\mathbb{N}}$.

If x is writable (eventually writable, accidentally writable) in every oracle $y \in A$, then x is writable (eventually writable, accidentally writable).

Infinite time register machines

An infinite time register machine (ITRM) stores integers in finitely many registers and works in ordinal time.

Theorem

Suppose that x is a real and A is a set of positive measure such that x is ITRM-computable from all $y \in A$. Then x is ITRM-computable.

Consider the variant of ordinal time/tape Turing machine whose time is bounded by an ordinal α.

Theorem

There are unboundedly many countable admissible ordinals α such that every real x which is α-computable from all elements of a set A of positive measure is α-computable.

Question

Are there analogous results for other ideals, such as the ideals associated to perfect set forcing or the forcing for adding a dominating function?