Infinite computations with random oracles

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Motivation

Many mathematical objects cannot be coded by integers, yet we can perform infinitary constructions with these objects

- constructing the algebraic closure of a field
- constructing the levels $L_\alpha$ of the constructible universe $L$

This motivates the study of infinitary computations, which give a precise meaning to various intuitive infinitary constructions.
Infinite time computations

Hamkins, Welch, Koepke and others studied Turing programs with infinite hardware and infinite time.

- analogies to Turing computability
- halting times
- relation with $\Pi^1_1$ and $\Sigma^1_2$ sets

Goals

- analogies to algorithmic randomness
- computability from a set of real oracles of positive measure
Infinite time Turing machines

Consider a Turing program which runs on the hardware of a Turing machine, but with infinite time (ITTM, Hamkins-Kidder 2000).

- the tape is a Turing tape
- the time is the ordinals ($Ord$)

The machine works as follows.

- the state is the $\liminf$ of the states at previous times
- the head moves to the $\liminf$ of its previous positions if this is finite, and to 0 otherwise
- the contents of each cell is the $\liminf$ of the contents at previous times

Example

Test if a symbol occurs infinitely often.
Test if a tree has an infinite branch.
Ordinal time/tape Turing machines

Consider a Turing program which runs on infinite hardware (OTM, Koepke 2006).

- the tape has length the ordinals
- the time is the ordinals

The transition in limit times is defined as follows.

- the state is the $\text{liminf}$ of the states at previous times
- the head moves to the $\text{liminf}$ of its previous positions
- the contents of each cell is the $\text{liminf}$ of the contents at previous times

Example

Add ordinals. $\alpha + \beta$ is defined by

- $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
- $\alpha + \lambda = \sup_{\beta < \lambda} \alpha + \beta$ for limits $\lambda$

We represent $\alpha$ by a symbol in place $\alpha$. 
# Computations

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Computations from many oracles

**Lemma (de Leuw-Moore-Shannon-Shapiro, Sacks)**

If $A \subseteq 2^\mathbb{N}$, $\mu(A) > 0$, and $x \in 2^\mathbb{N}$ is computable from all $y \in A$, then $x$ is computable.

**Proof.**

There is an interval $U_s$ with $\frac{\mu(U_s \cap A)}{\mu(U_s)} > 0.5$ by the Lebesgue density theorem. Assume $\mu(A) > 0.5$.

Each bit is computed from some $s_0, \ldots, s_n$ with $\mu(U_{s_0} \cup \ldots \cup U_{s_n}) > 0.5$.

Is this true for infinite computations?

- the machine can read all input bits during a computation
- we cannot list all possible input words
Computable sets

Definition

Let \( x, y \in 2^\mathbb{N} \), \( A \subseteq 2^\mathbb{N} \).

- \( x \) is *OTM-computable* from \( y \) (\( x \leq_{OTM} y \)) if there is an *OTM* \( P \) such that \( P \) halts on input \( y \) with output \( x \) (\( P^y = x \)).
- \( A \) is *OTM-computable* if there is an *OTM* \( P \) such that \( P \) halts on all inputs \( x \in 2^\mathbb{N} \), and \( x \in A \) iff \( P^x = 0 \).

- ITTMs compute \( \Pi^1_1 \) and \( \Sigma^1_1 \) sets of reals (Hamkins-Lewis)
- OTMs compute \( \Delta^1_2 \) sets of reals (Koepke-Seyfferth)
- OTMs with ordinal oracles compute \( L \) (Koepke)
The computable reals are those in some $L_\alpha$ for various machines.

**Definition**

$L_0 := \emptyset$

$L_{\alpha+1} := \text{Def}(L_\alpha, \in) := \{ X \subseteq L_\alpha \mid X = \{ x \in L_\alpha \mid (L_\alpha, \in) \models \varphi(x, a) \} \} $ for some $a \in L_\alpha$ and some first order formula $\varphi$

$L := \bigcup_{\alpha \in \text{Ord}} L_\alpha$
Halting times

Definition
Let $\eta^x$ denote the supremum of halting times of $OTMs$ with oracle $x$.

Lemma
The following conditions are equivalent for reals $x, y$.
\begin{itemize}
  \item $x$ is $\Delta^1_2$ in $y$
  \item $x \leq_{OTM} y$
  \item $x \in L_{\eta^y}[y]$
\end{itemize}
OTM computations in $L$

**Theorem**

Suppose that $V = L$. There is a real $x$ and a co-countable set $A \subseteq 2^\mathbb{N}$ such that

- $x$ is OTM-computable from every $y \in A$ but
- $x$ is not OTM-computable.

**Corollary**

Assume that $V = L$.

- Let $z$ denote the halting problem for OTMs. Then $z \leq_{OTM} x$ for every non-OTM-computable real $x$.
- For all reals $x$ and $y$, $x \leq_{OTM} y$ or $y \leq_{OTM} x$. 
Cohen and random reals

Definition
Suppose that $x \in 2^\mathbb{N}$.

- $x$ is Cohen over $L_\alpha$ if $x \in B$ for every comeager Borel set $B$ with a Borel code in $L_\alpha$.
- $x$ is random over $L_\alpha$ if $x \in B$ for every measure 1 Borel set $B$ with a Borel code in $L_\alpha$.

- related to forcing in set theory
- related to randomness in computability
Theorem

- Suppose that for every \( x \in 2^\mathbb{N} \), the set of random reals over \( L[x] \) has measure 1 (iff every \( \Sigma^1_2 \) set is Lebesgue measurable).

  If \( A \subseteq 2^\mathbb{N} \) has positive measure and \( x \) is OTM-computable from every \( y \in A \), then \( x \) is OTM-computable.

- Suppose that for every \( x \in 2^\mathbb{N} \), the set of Cohen reals over \( L[x] \) is comeager (iff every \( \Sigma^1_2 \) set has the property of Baire).

  If \( A \) is a nonmeager set with the property of Baire and \( x \) is OTM-computable from every \( y \in A \), then \( x \) is OTM-computable.
OTM computations with ordinal parameters

**Lemma**

A real $x$ is OTM-computable from $y$ with ordinal oracles iff $x \in L[y]$, i.e. $x$ is constructible from $y$.

In $L$, and in any model in which $(2^\mathbb{N})^L$ is not Lebesgue measurable, our question is trivial.

The following result follows easily from work of Judah-Shelah.

**Theorem**

There is a forcing $\mathbb{P}$ in $L$ such that in any $\mathbb{P}$-generic extension of $L$ there is a measure 1 set $A \subseteq 2^\mathbb{N}$ and

- every $x \in A$ can be constructed from every $y \in A$
- $A$ contains no constructible real
- $(2^\mathbb{N})^L$ has measure 0
OTM computations with ordinal parameters

**Theorem**

- Suppose that for every real $x$, there is a random real over $L[x]$. 
  If $A$ has positive measure and $x \in 2^\mathbb{N}$ is constructible from each $y \in A$, then $x \in L$.

- Suppose that for every real $x$, there is a Cohen real over $L[x]$. 
  If $A$ is a nonmeager Borel set and $x \in 2^\mathbb{N}$ is constructible from each $y \in A$, then $x \in L$.

**Question**

Is it consistent that there is a nonconstructible real $x$ and a Borel set $A$ of measure 1 such that $x$ is OTM-computable without parameters from every $y \in A$?
ITTM writable reals

Definition
Suppose that $x \in 2^\mathbb{N}$.

- Let $\lambda^x$ denote the supremum of ITTM-writable ordinals (write-halt) with oracle $x$.
- Let $\zeta^x$ denote the supremum of ITTM-eventually writable ordinals (write-keep) with oracle $x$.
- Let $\Sigma^x$ denote the supremum of ITTM-accidentally writable ordinals (write) with oracle $x$.

Theorem (Welch)

- The reals writable (eventually writable, accidentally writable) in the oracle $x$ are exactly those in $L_{\lambda^x}[x]$ ($L_{\zeta^x}[x]$, $L_{\Sigma^x}[x]$).
- $\zeta^x, \Sigma^x$ is the lexically least pair of ordinals with $L_{\zeta^x}[x] \prec_2 L_{\Sigma^x}[x]$.
- $\lambda^x$ is minimal with $L_{\lambda^x}[x] \prec_1 L_{\zeta^x}[x]$. 

ITTM computations from many oracles

Lemma
If $x$ is Cohen generic over $L_{\Sigma+1}$ then
\begin{itemize}
  \item $L_\lambda[x] \prec_{\Sigma_1} L_\zeta[x] \prec_{\Sigma_2} L_\Sigma[x]$.
  \item $\lambda^x = \lambda$, $\zeta^x = \zeta$ and $\Sigma^x = \Sigma$.
\end{itemize}

Theorem
Suppose that $A \subseteq 2^\mathbb{N}$ is a nonmeager Borel set and $x \in 2^\mathbb{N}$.

If $x$ is writable (eventually writable, accidentally writable) in every oracle $y \in A$, then $x$ is writable (eventually writable, accidentally writable).

Conjecture
Suppose that $A \subseteq 2^\mathbb{N}$ is a set with positive measure and $x \in 2^\mathbb{N}$.

If $x$ is writable (eventually writable, accidentally writable) in every oracle $y \in A$, then $x$ is writable (eventually writable, accidentally writable).
Infinite time register machines

An infinite time register machine (ITRM) stores integers in finitely many registers and works in ordinal time.

Theorem
Suppose that $x$ is a real and $A$ is a set of positive measure such that $x$ is ITRM-computable from all $y \in A$. Then $x$ is ITRM-computable.

Consider the variant of ordinal time/tape Turing machine whose time is bounded by an ordinal $\alpha$.

Theorem
There are unboundedly many countable admissible ordinals $\alpha$ such that every real $x$ which is $\alpha$-computable from all elements of a set $A$ of positive measure is $\alpha$-computable.
Question

Are there analogous results for other ideals, such as the ideals associated to perfect set forcing or the forcing for adding a dominating function?