# Sets Have Simple Members[1]

Samuel Epstein
Boston University
(Joint work with Leonid A. Levin)

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National University of Singapore
Institute for Mathematical Science

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# Overview

- Sets Have Simple Members:

  Incompleteness meets Occam's Razor
- New General Proof Technique:

  Separate Enumeration and Combinatorics
- Algorithmic Foundations of Quantum Mechanics

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- Sets Have Simple Members:

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- New General Proof Technique:

  Separate Enumeration and Combinatorics

- Algorithmic Foundations of Quantum Mechanics
  - Quantum Chain Rule
  - Random Measurements
  - Generalized No Communication Theorem
  - Equivalences Between Quantum Entropies

# Mutual Information with Halting Sequence

- Leverage the term: $\mathbf{I}(x;\mathcal{H}) = \mathbf{K}(x) - \mathbf{K}(x|\mathcal{H})$.
- Information non growth: we have $\mathbf{I}(A(x);\mathcal{H}) <^{+} \mathbf{I}(x;\mathcal{H})$.

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- No Go Theorems
  - If string $x$ has high value $\mathcal{P}$ then $\mathbf{I}(x;\mathcal{H})$ is high.
  - Therefore is no algorithm that can produce $x$ with high $\mathcal{P}$.

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  - If string $x$ has high value $\mathcal{P}$ then $\mathbf{I}(x;\mathcal{H})$ is high.
  - Therefore is no algorithm that can produce $x$ with high $\mathcal{P}$.
- Any total extension $U : \{0,1\}^* \to \{0,1\}$ of first $2^n$ inputs of universal partial predicate $u : \{0,1\}^* \to \{0,1\}$ has $n <^{\log} \mathbf{I}(U;\mathcal{H})$.

# “Finitize” Theorems

## Theorem

- $\mathbf{K}(x)$ *is not recursive.*

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## Problem (June 19th)

- *What is* $\mathbf{I}(\gamma;\mathcal{H})$ *for a set* $\gamma$ *of* $n$ *strings containing* $k$ *entries* $\langle x, [x \textit{ is random}]\rangle$.

## Problem (June 20th)

*What properties does* $\mathbf{I}(x;\emptyset'')$ *have?*

# Sets have Simple Members

### Definition (Prior of a Set)

For a set $D$, we have $\mathbf{m}(D) = \sum_{x\in D} \mathbf{m}(x)$.

### Theorem

$\min_{x\in D} -\log \mathbf{m}(x) <^{\log} -\log \mathbf{m}(D) + \mathbf{I}(D;\mathcal{H})$.

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The prior of natural sets are dominated by its simplest element.

# All Sampling Methods have Outliers

### Definition (Deficiency of Randomness)

For computable measure $P$, we have:
$\mathbf{d}(a|P) = -\log P(a) - \mathbf{K}(a)$.

### Theorem

$\log \|D\| <^{\log} \max_{a\in D} \mathbf{d}(a|P) + \mathbf{I}(D;\mathcal{H})$.

All natural samples $D$ of size $2^n$ have an outlier $x\in D$ with score $n$.

# New Proof Technique

- Separate enumerative and combinatorial arguments.

1. Start with definitions
2. Make everything computable $\mathbf{m} \to \mathbf{m}'$.
3. Perform combinatorics
4. Convert back $\mathbf{m}' \to \mathbf{m}$. (Error term $\mathbf{I}(;\mathcal{H})$).

# Quantum Results

- Generalize randomness notions from Cantor space $\Omega$ to Hilbert space $\mathcal{H}_n$ of $n$ qubits.
- Use Gács entropy-2 of quantum state $|\psi\rangle$.
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**Theorem (Chain Rule Inequality)**

$\mathbf{H}(|\psi\rangle) + \mathbf{H}(|\phi\rangle|\mathrm{Encode}(\psi)) <^{\log} \mathbf{H}(|\psi\rangle|\phi\rangle)$.

**Theorem (Relation between Entropies)**

$\mathbf{H}(|\psi\rangle) <^{\log} \mathrm{Kq}(|\psi\rangle) \leq \mathbf{H}(|\psi\rangle) + \mathbf{I}(|\psi\rangle;\mathcal{H})$.
$\mathbf{H}(|\psi\rangle) <^{\log} QC(|\psi\rangle)$.

# Thank You

Kolmogorov (left) delivers a talk at a Soviet information theory symposium. (Tallinn, 1973).