Ornstein Isomorphism and Algorithmic Randomness

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April 1, 2014

Abstract

In 1970, Donald Ornstein proved a landmark result in dynamical systems, *viz.*, two Bernoulli systems with the same entropy are isomorphic except for a measure 0 set [8]. Keane and Smorodinsky [6] gave a finitary proof of this result. They also indicated how one can generalize the result to mixing Markov Shifts in [5]. We adapt the construction given in [6] to show that if two computable mixing Markov systems have the same entropy, then there is a layerwise computable isomorphism defined on all Martin-Löf random points in the system. Since the set of Martin-Löf random points forms a measure 1 set, it implies the classical result for such systems.

This result uses several recent developments in computable analysis and algorithmic randomness. Following the work by Braverman [1], Nandakumar [7], and Hoyrup and Rojas [3] introduced discontinuous functions into the study of algorithmic randomness. We utilize Hoyrup and Rojas' elegant notion of layerwise computable functions to produce the test of randomness in our result. Further, we use the recent result of the effective Shannon-McMillan-Breiman theorem, independently established by Hochman [2] and Hoyrup [4] to prove the properties of our construction.

We show that the result cannot be improved to include all points in the systems - only trivial computable isomorphisms exist between systems with the same entropy.

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