Triviality within and beyond Hyperarithmetic

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If a real computes a function which is not dominated by any computable function, it also computes a weakly $\mathbf{1}$-generic real.

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However, the negation of the above property is consistent in set theory in the sense that

if G is a Laver or Miller generic over $M \models \mathbf{ZFC}$,
then $M[G]$ has a function not dominated by any M-function,
whereas $M[G]$ contains no real Cohen over M.

Indeed, the so-called Laver property implies the failure of a much weaker property that

every unbounded real contains an information of a nontrivial real

in the sense that

if G is a Laver or Miller generic over $M \models \mathbf{ZFC}$,
then $M[G]$ has a function not dominated by any M-function,
whereas $M[G]$ contains only M-trivial reals.

Here, a real $x \in 2^\omega$ is M-trivial if
for every partial prefix-free function $\varphi :\subseteq 2^{<\omega} \to 2^{<\omega}$ in M
there exists a partial prefix-free function $\psi :\subseteq 2^{<\omega} \to 2^{<\omega}$ in M
such that $K_\psi(x \upharpoonright n) \leq K_\varphi(n) + O(1)$.

The Laver property is a key notion in the proof of Richard Laver's theorem (1976):

"the Borel conjecture is independent of ZFC"

where the Borel conjecture (Emile Borel 1919) states that

"every strong measure zero set $X \subseteq \mathbb{R}$ is countable".

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"every strong measure zero set $X \subseteq \mathbb{R}$ is countable".

A notion $\mathbb{P}$ of forcing satisfies *the Laver property* if for every $\mathbb{P}$-name $\dot{g}$ and for every function $f \in \omega^\omega$ in the ground model, if

$$\Vdash_{\mathbb{P}} \dot{g} \in \omega^\omega \;\&\; (\forall n \in \omega)\; \dot{g}(n) < f(n),$$

then there exists *a sequence* $\{T_n\}_{n\in\omega} \in ([\omega]^{<\omega})^\omega$ *with* $|T_n| \leq 2^n$ *in the ground model* such that

$$\Vdash_{\mathbb{P}} (\forall n \in \omega)\; \dot{g}(n) \in T_n.$$

countable $\Rightarrow$ null-additive

$\Rightarrow$ $\mathcal{E}$-additive $\Leftrightarrow$ meager-additive

$\Rightarrow$ strong measure zero

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Theorem (K. and Miyabe)

1. effectively null-additive $=$ **uni-Low(SR)** $=$ Schnorr trivial.

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Theorem (K. and Miyabe)

1. effectively null-additive = **uni-Low(SR)** = Schnorr trivial.
2. effectively $\mathcal{E}$-additive = **uni-Low(WR)** = effectively meager-additive = **uni-Low(W1G)**.

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1. effectively null-additive $=$ $\mathbf{uni\text{-}Low(SR)}$ $=$ Schnorr trivial.
2. effectively $\mathcal{E}$-additive $=$ $\mathbf{uni\text{-}Low(WR)}$ $=$ effectively meager-additive $=$ $\mathbf{uni\text{-}Low(W1G)}$.
3. effectively strong measure zero $=$ $\mathbf{uni\text{-}Low(WR, SR)}$.

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Theorem (K. and Miyabe)

1. Δ^1_1-null-additive $=$ $\mathbf{uni\text{-}Low}(\Delta^1_1\mathbf{R})$ $=$ Δ^1_1-trivial.
2. Δ^1_1-$\mathcal{E}$-additive $=$ $\mathbf{uni\text{-}Low}(\Delta^1_1\mathbf{WR})$
   $=$ Δ^1_1-meager-additive $=$ $\mathbf{uni\text{-}Low}(\Delta^1_1\mathbf{G})$.
3. Δ^1_1-strong measure zero $=$ $\mathbf{uni\text{-}Low}(\Delta^1_1\mathbf{WR}, \Delta^1_1\mathbf{R})$.

Main Theorem

There is a real $x \in 2^\omega$ such that

1. x has a minimal hyperdegree,
2. there is a function $f \leq_h x$ not dominated by any Δ^1_1 function (hence, x is neither **Low(Δ^1_1WR)** nor **Low(Δ^1_1R)**),
3. every real $y \leq_h x$ is Δ^1_1-trivial (hence, x is **uni-Low(Δ^1_1R)**),
4. and x is **Low(Δ^1_1R, Δ^1_1WR)**.

Our main theorem will be proved by using rational perfect forcing over the ω_1^{CK}-th rank of Gödel's constructible universe.

Indeed, for any $M \models \mathbf{KP}$, we can show an "almost" same property by using rational perfect forcing over M; where

if Γ is a Spector pointclass ($M_\Gamma \models \mathbf{KP}$ is the companion of Γ), then we may naturally introduce a reducibility notion $\leq_\Delta$, and the least non-Δ-computable ordinal λ_Γ since Γ is normed.

However, the main difficulty is that:

- This forcing is not a set forcing over M_Γ.
- It is not clear whether a generic real preserves the ordinal λ_Γ.

At least, we can overcome this difficulty for:

- $\Gamma = \Pi^1_1$,
- $\Gamma =$"$\mathbf{E_n}$-computably enumerable", or
- $\Gamma = \Sigma^1_{2n}, \Pi^1_{2n+1}$ (under projective determinacy)

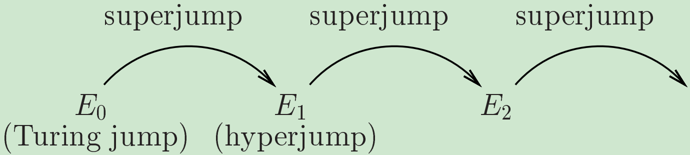
by some known "ad-hoc" arguments by G. Sacks, J. Shinoda, and A. Kechris.

A normal type **3** functional $\mathrm{sJ} : [\mathbb{N}^{\mathbb{N}} \to \mathbb{N}] \to (\mathbb{N} \times \mathbb{N}^{\mathbb{N}}) \to 2$ (*superjump operator*, 1959) is defined as follows:
for any type **2** functional F,

$$\mathrm{sJ}(F)(e, x) = \begin{cases} 1 & \text{if } \Phi_e^F(x) \downarrow, \\ 0 & \text{if } \Phi_e^F(x) \uparrow \end{cases}$$

Here, Φ_e^F is the e-th computation relative to the functional F in the sense of Kleene's finite type computability (S1-S9).

Define $\mathrm{E}_0 := {}^2\mathrm{E}$, and $\mathrm{E}_{n+1} := \mathrm{sJ}(\mathrm{E}_n)$.



$$\omega < \omega_1^{\mathrm{CK}} < \omega_1^{\mathrm{E}_1} < \omega_1^{\mathrm{E}_2} < \cdots < \omega_1^{\mathrm{E}_\omega} < \cdots < \omega_1^{\mathrm{sJ}} < \lambda < \zeta < \Sigma < \delta_2^1 < \aleph_1$$

Suppose that

- $\Gamma = \Pi^1_1$,
- $\Gamma =$"$\mathbf{E_n}$-computably enumerable", or
- $\Gamma = \Game\Gamma'$ is a $\Game$-generated reflecting Specter pointclass satisfying $\mathbf{Det}(\mathbf{Borel}(\Gamma'))$,
  (in particular, Γ can be Σ^1_{2n} or Π^1_{2n+1} under projective determinacy)

Main Theorem

There is a real $x \in 2^\omega$ such that

1. x has a minimal Δ-degree,
2. there is a function $f \leq_\Delta x$ not dominated by any Δ function (hence, x is neither **Low(**Δ-coded **WR)** nor **Low(**Δ-coded **R)**),
3. every real $y \leq_\Delta x$ is Δ-trivial (hence, x is **uni-Low(**Δ-coded **R)**),
4. and x is **Low(**Δ-coded **R,** Δ-coded **WR)**.

What's rational perfect forcing **PT**?

- Each forcing condition is a superperfect $\mathbf{\Delta}^1_1$-subtree of $\omega^{<\omega}$, that is, $T \subseteq \omega^{<\omega}$ is $\mathbf{\Delta}^1_1$, and every $\sigma \in T$ has an extension $\tau \in T$ having infinitely many immediate successors.
- ordered by inclusion.

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- **PT** has the continuous reading of names (an abstract analog of N. Luzin's theorem for any forcing notion).
- **PT** has the one-to-one or constant property (hence, every generic real has a minimal hyperdegree).
- **PT** has the Laver property (hence, every generic real hyp-computes only trivial reals).

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Surprisingly, B. Monin recently announced that

every $\mathbf{\Delta}^1_1$-dominant hyp-computes a $\mathbf{\Delta}^1_1$-generic/random real.

Consequently, Laver forcing **LT** does not work over $L_{\omega_1^{\mathrm{CK}}}$.

Question

Does Laver forcing **LT** at the $\mathbf{E_n}$-level, the $\mathbf{\Delta}^1_2$-level, or the **ITTM**-level work well?

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Question

- Can we separate “[partial continuous]-lowness for $\mathbf{\Delta}^1_1$-randomness” and “lowness for $\mathbf{\Delta}^1_1$-randomness”?
- Is there a proper hierarchy of “[Baire α]-lowness for $\mathbf{\Delta}^1_1$-randomness”?