Degrees containing no members of thin classes

Wu Guohua

School of Physical and Mathematical Sciences

Nanyang Technological University

CCR 2014

Density Theorems

- Sacks Density Theorem
  - Sacks Coding Strategy
  - Sacks Preservation Strategy

Density Theorems

- Sacks Density Theorem
  - Sacks Coding Strategy
  - Sacks Preservation Strategy

- Fejer's Density Theorem (nonbranching degrees)

Density Theorems

- Sacks Density Theorem
  - Sacks Coding Strategy
  - Sacks Preservation Strategy
- Fejer's Density Theorem (nonbranching degrees)
- Slaman's Density Theorem (branching degrees)

Density Theorems

- Sacks Density Theorem
  - Sacks Coding Strategy
  - Sacks Preservation Strategy
- Fejer's Density Theorem (nonbranching degrees)
- Slaman's Density Theorem (branching degrees)
- Ladner-Sasso's Splitting Theorem (c.e. wtt-degrees)

Density Theorems

- Sacks Density Theorem
  - Sacks Coding Strategy
  - Sacks Preservation Strategy
- Fejer's Density Theorem (nonbranching degrees)
- Slaman's Density Theorem (branching degrees)
- Ladner-Sasso's Splitting Theorem (c.e. wtt-degrees)
- Harrington's Splitting Theorem (low_2 c.e. degrees)

Intervals of Nonhemimaximal Degrees - a nondensity theorem

- Hemimaximal sets (jump inversion) and Nonhemimaximal degrees

Intervals of Nonhemimaximal Degrees - a nondensity theorem

- Hemimaximal sets (jump inversion) and Nonhemimaximal degrees
- Intervals of nonhemimaximal degrees

Π^0_1-Classes

- We are talking about Π^0_1-classes of the Cantor space.
- Jockusch-Soare: Low basis theorem
- Grozek-Slaman:

  There exists a nonempty Π^0_1-class with all members bounding minimal degrees.

Minimal Π^0_1-Classes and Thin Π^0_1-Classes

- A Π^0_1-class P is *minimal* if for every Π^0_1-subclass Q, either Q is finite or $P \setminus Q$ is finite.
- A Π^0_1-class P is *thin* if every Π^0_1-subclass Q of P is clopen in P.

Minimal Π^0_1-Classes and Thin Π^0_1-Classes

- A Π^0_1-class P is *minimal* if for every Π^0_1-subclass Q, either Q is finite or $P \setminus Q$ is finite.
- A Π^0_1-class P is *thin* if every Π^0_1-subclass Q of P is clopen in P.

Theorem [CDJS]

Let P be a Π^0_1-class. Then

- If P is minimal and has an incomputable member, then P is thin.
- If P is thin and the Cantor-Bendixson derivative $D(P)$ is a singleton, then P is minimal.

Degrees of members of thin Π^0_1-classes

Theorem [CDJS]

Let P be a thin Π^0_1-class and $A \in P$. Then $A' \leq_T A \oplus \emptyset''$.

Thus no thin Π^0_1-class can have a member $A \geq_T \emptyset''$.

Degrees of members of thin Π^0_1-classes

Theorem [CDJS]

Let P be a thin Π^0_1-class and $A \in P$. Then $A' \leq_T A \oplus \emptyset''$.

Thus no thin Π^0_1-class can have a member $A \geq_T \emptyset''$.

Theorem [CDJS]

$\mathbf{0}'$ contains a Π^0_1 set as a member of a thin Π^0_1-class.

Degrees of members of thin Π^0_1-classes

Theorem [CDJS]

Let P be a thin Π^0_1-class and $A \in P$. Then $A' \leq_T A \oplus \emptyset''$.

Thus no thin Π^0_1-class can have a member $A \geq_T \emptyset''$.

Theorem [CDJS]

$\mathbf{0}'$ contains a Π^0_1 set as a member of a thin Π^0_1-class.

Theorem [CDJS] - a weak density

For any c.e. sets $A <_T C$, there is a set B with $A \leq_T B \leq_T C$ and B is a member of a minimal (and hence thin) Π^0_1-class.

Degrees containing no members of thin Π^0_1-classes

Theorem [CDJS]

There is a c.e. set C such that no set $A \equiv_T C$ belongs to any thin Π^0_1-class.

Degrees containing no members of thin Π^0_1-classes

Theorem [CDJS]

There is a c.e. set C such that no set $A \equiv_T C$ belongs to any thin Π^0_1-class.

Observation [DWY]

Degrees containing members of thin classes are not closed under join.

CDJS's Construction of a *NONTHIN* degree

Construct a single set C such that the degree of C does not contain any member of thin class.

CDJS's Construction of a *NONTHIN* degree

Construct a single set C such that the degree of C does not contain any member of thin class.

C is constructed to meet the following requirements:

$\mathcal{R}_e$: if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, then

CDJS's Construction of a *NONTHIN* degree

Construct a single set C such that the degree of C does not contain any member of thin class.

C is constructed to meet the following requirements:

$\mathcal{R}_e$: if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, then

- either $C \neq \Psi_e(\Phi_e(C))$, or
- $\Phi_e(C)$ is not in $[T_e]$, or
- $[T_e]$ is not thin.

CDJS's Construction of a *NONTHIN* degree

Construct a single set C such that the degree of C does not contain any member of thin class.

C is constructed to meet the following requirements:

$\mathcal{R}_e$: if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, then

- either $C \neq \Psi_e(\Phi_e(C))$, or
- $\Phi_e(C)$ is not in $[T_e]$, or
- $[T_e]$ is not thin.

We will construct a subtree S_e of T_e such that if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, with $C = \Psi_e(\Phi_e(C))$, and $\Phi_e(C)$ a branch in $[T_e]$, then $[S_e]$ witnesses that $[T_e]$ is not thin.

CDJS's Construction of a *NONTHIN* degree

Construct a single set C such that the degree of C does not contain any member of thin class.

C is constructed to meet the following requirements:

$\mathcal{R}_e$: if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, then

- either $C \neq \Psi_e(\Phi_e(C))$, or
- $\Phi_e(C)$ is not in $[T_e]$, or
- $[T_e]$ is not thin.

We will construct a subtree S_e of T_e such that if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, with $C = \Psi_e(\Phi_e(C))$, and $\Phi_e(C)$ a branch in $[T_e]$, then $[S_e]$ witnesses that $[T_e]$ is not thin.

- Compare it with the construction of nonhemimaximal degrees.

Subrequirements

$\mathcal{R}_{e,i}$: There exist an interval $(x_{e,i}, z_{e,i})$ such that $[T_e]$ contains a branch extending $\Phi_e(C) \upharpoonright x_{e,i}$, but not $\Phi_e(C) \upharpoonright z_{e,i}$.

- If i is even, then all nodes in T_e extending $\Phi_e(C) \upharpoonright x_{e,i}$, but not $\Phi_e(C) \upharpoonright z_{e,i}$, will be *put* on S_e.
- If i is odd, then all nodes in T_e extending $\Phi_e(C) \upharpoonright x_{e,i}$, but not $\Phi_e(C) \upharpoonright z_{e,i}$, will be *terminated* on S_e.

So, if all the $\mathcal{R}_{e,i}$-subrequirements are satisfied, then $[S_e]$ is a subclass of $[T_e]$, containing, and also missing, infinitely many branches of $[T_e]$.

Subrequirements

$\mathcal{R}_{e,i}$: There exist an interval $(x_{e,i}, z_{e,i})$ such that $[T_e]$ contains a branch extending $\Phi_e(C) \upharpoonright x_{e,i}$, but not $\Phi_e(C) \upharpoonright z_{e,i}$.

- If i is even, then all nodes in T_e extending $\Phi_e(C) \upharpoonright x_{e,i}$, but not $\Phi_e(C) \upharpoonright z_{e,i}$, will be *put* on S_e.
- If i is odd, then all nodes in T_e extending $\Phi_e(C) \upharpoonright x_{e,i}$, but not $\Phi_e(C) \upharpoonright z_{e,i}$, will be *terminated* on S_e.

So, if all the $\mathcal{R}_{e,i}$-subrequirements are satisfied, then $[S_e]$ is a subclass of $[T_e]$, containing, and also missing, infinitely many branches of $[T_e]$.

- The crucial point is after one region is terminated, $\Phi_e(C)$ will never come back to this region again, in the remainder of the construction.

Subrequirements

$\mathcal{R}_{e,i}$: There exist an interval $(x_{e,i}, z_{e,i})$ such that $[T_e]$ contains a branch extending $\Phi_e(C) \upharpoonright x_{e,i}$, but not $\Phi_e(C) \upharpoonright z_{e,i}$.

- If i is even, then all nodes in T_e extending $\Phi_e(C) \upharpoonright x_{e,i}$, but not $\Phi_e(C) \upharpoonright z_{e,i}$, will be *put* on S_e.
- If i is odd, then all nodes in T_e extending $\Phi_e(C) \upharpoonright x_{e,i}$, but not $\Phi_e(C) \upharpoonright z_{e,i}$, will be *terminated* on S_e.

So, if all the $\mathcal{R}_{e,i}$-subrequirements are satisfied, then $[S_e]$ is a subclass of $[T_e]$, containing, and also missing, infinitely many branches of $[T_e]$.

- The crucial point is after one region is terminated, $\Phi_e(C)$ will never come back to this region again, in the remainder of the construction.
- Threading strategy, for the consistency between $\mathcal{R}$-strategies.

Upwards Density of NONTHIN degrees

A joint work with Downey and Yang.

Upwards Density of NONTHIN degrees

A joint work with Downey and Yang.

- Fix B as an incomplete c.e. set.

Construct a c.e. set C such that the degree of $B \oplus C$ does not contain any member of thin classes.

Upwards Density of NONTHIN degrees

A joint work with Downey and Yang.

- Fix B as an incomplete c.e. set.

Construct a c.e. set C such that the degree of $B \oplus C$ does not contain any member of thin classes.

If $B \oplus C$ has NONTHIN degree, then it is incomplete, as CDJS already proved that $\mathbf{0}'$ contains elements of thin classes.

Requirements

C is constructed to meet the following requirements:

$\mathcal{R}_e$: if $\Phi_e(B \oplus C)$, $\Psi_e(\Phi_e(B \oplus C))$ and $\Theta_e(\Phi_e(B \oplus C))$ are all total, then

- either $C \neq \Psi_e(\Phi_e(B \oplus C))$, or
- $B \neq \Theta_e(\Phi_e(B \oplus C))$, or
- $\Phi_e(B \oplus C)$ is not in $[T_e]$, or
- $[T_e]$ is not thin.

Requirements

C is constructed to meet the following requirements:

$\mathcal{R}_e$: if $\Phi_e(B \oplus C)$, $\Psi_e(\Phi_e(B \oplus C))$ and $\Theta_e(\Phi_e(B \oplus C))$ are all total, then

- either $C \neq \Psi_e(\Phi_e(B \oplus C))$, or
- $B \neq \Theta_e(\Phi_e(B \oplus C))$, or
- $\Phi_e(B \oplus C)$ is not in $[T_e]$, or
- $[T_e]$ is not thin.

Concerns:

Requirements

C is constructed to meet the following requirements:

$\mathcal{R}_e$: if $\Phi_e(B \oplus C)$, $\Psi_e(\Phi_e(B \oplus C))$ and $\Theta_e(\Phi_e(B \oplus C))$ are all total, then

- either $C \neq \Psi_e(\Phi_e(B \oplus C))$, or
- $B \neq \Theta_e(\Phi_e(B \oplus C))$, or
- $\Phi_e(B \oplus C)$ is not in $[T_e]$, or
- $[T_e]$ is not thin.

Concerns:

- Divergence
- Threading strategy
- B's change can bring $\Phi_e(B \oplus C)$ into a terminated region. What shall we do?

Full density: True.

Full density: True.

Thanks!