Degrees containing no members of thin classes

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Density Theorems

- Sacks Density Theorem
  - Sacks Coding Strategy
  - Sacks Preservation Strategy
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- Fejer’s Density Theorem (nonbranching degrees)
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- Ladner-Sasso’s Splitting Theorem (c.e. wtt-degrees)
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- Harrington’s Splitting Theorem (low₂ c.e. degrees)
Intervals of Nonhemimaximal Degrees - a nondensity theorem

- Hemimaximal sets (jump inversion) and Nonhemimaximal degrees
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- Hemimaximal sets (jump inversion) and Nonhemimaximal degrees
- Intervals of nonhemimaximal degrees
We are talking about $\Pi^0_1$-classes of the Cantor space.

Jockusch-Soare: Low basis theorem

Grozek-Slaman:

There exists a nonempty $\Pi^0_1$-class with all members bounding minimal degrees.
Minimal $\Pi^0_1$-Classes and Thin $\Pi^0_1$-Classes

- A $\Pi^0_1$-class $P$ is *minimal* if for every $\Pi^0_1$-subclass $Q$, either $Q$ is finite or $P \setminus Q$ is finite.

- A $\Pi^0_1$-class $P$ is *thin* if every $\Pi^0_1$-subclass $Q$ of $P$ is clopen in $P$. 

Theorem [CDJS] Let $P$ be a $\Pi^0_1$-class. Then

- If $P$ is minimal and has an incomputable member, then $P$ is thin.

- If $P$ is thin and the Cantor-Bendixson derivative $D(P)$ is a singleton, then $P$ is minimal.
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Degrees of members of thin $\Pi^0_1$-classes

Theorem [CDJS]
Let $P$ be a thin $\Pi^0_1$-class and $A \in P$. Then $A' \leq_T A \oplus \phi''$. Thus no thin $\Pi^0_1$-class can have a member $A \geq_T \phi''$. 
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Theorem [CDJS]
Let $P$ be a thin $\Pi^0_1$-class and $A \in P$. Then $A' \leq_T A \oplus \emptyset''$.

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Theorem [CDJS]
$0'$ contains a $\Pi^0_1$ set as a member of a thin $\Pi^0_1$-class.
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**Theorem [CDJS]**
$0'$ contains a $\Pi^0_1$ set as a member of a thin $\Pi^0_1$-class.

**Theorem [CDJS] - a weak density**
For any c.e. sets $A \lt_T C$, there is a set $B$ with $A \leq_T B \leq_T C$ and $B$ is a member of a minimal (and hence thin) $\Pi^0_1$-class.
Degrees containing no members of thin $\Pi^0_1$-classes

**Theorem [CDJS]**

There is a c.e. set $C$ such that no set $A \equiv_T C$ belongs to any thin $\Pi^0_1$-class.
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**Observation [DWY]**

Degrees containing members of thin classes are not closed under join.
CDJS’s Construction of a NONTHIN degree

Construct a single set $C$ such that the degree of $C$ does not contain any member of thin class.
CDJS’s Construction of a \textit{Nonthin} degree

Construct a single set $C$ such that the degree of $C$ does not contain any member of thin class.

$C$ is constructed to meet the following requirements:

$R_e$: if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, then

$\blacksquare$ either $C \neq \Psi_e(\Phi_e(C))$,

$\blacksquare$ $\Phi_e(C)$ is not in $[T]$,

$\blacksquare$ $[T]$ is not thin.

We will construct a subtree $S_e$ of $T$ such that if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, with $C = \Psi_e(\Phi_e(C))$, and $\Phi_e(C)$ a branch in $[T]$, then $[S_e]$ witnesses that $[T]$ is not thin.

$\blacksquare$ Compare it with the construction of nonhemimaximal degrees.
CDJS’s Construction of a *NONTIN* degree

Construct a single set $C$ such that the degree of $C$ does not contain any member of thin class.

$C$ is constructed to meet the following requirements:

$\mathcal{R}_e$: if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, then

- either $C \neq \Psi_e(\Phi_e(C))$, or
- $\Phi_e(C)$ is not in $[T_e]$, or
- $[T_e]$ is not thin.
CDJS’s Construction of a \textit{Nonthin} degree

Construct a single set $C$ such that the degree of $C$ does not contain any member of thin class.

$C$ is constructed to meet the following requirements:

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\quad \begin{align*}
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\text{either } & C \neq \Psi_e(\Phi_e(C)), \text{ or } \\
\text{either } & \Phi_e(C) \text{ is not in } [T_e], \text{ or } \\
\text{either } & [T_e] \text{ is not thin.}
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We will construct a subtree $S_e$ of $T_e$ such that if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, with $C = \Psi_e(\Phi_e(C))$, and $\Phi_e(C)$ a branch in $[T_e]$, then $[S_e]$ witnesses that $[T_e]$ is not thin.
CDJS’s Construction of a \textit{NORTHIN} degree

Construct a single set $C$ such that the degree of $C$ does not contain any member of thin class.

$C$ is constructed to meet the following requirements:

$\mathcal{R}_e$: if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, then

- either $C \neq \Psi_e(\Phi_e(C))$, or
- $\Phi_e(C)$ is not in $[T_e]$, or
- $[T_e]$ is not thin.

We will construct a subtree $S_e$ of $T_e$ such that if $\Phi_e(C)$ and $\Psi_e(\Phi_e(C))$ are both total, with $C = \Psi_e(\Phi_e(C))$, and $\Phi_e(C)$ a branch in $[T_e]$, then $[S_e]$ witnesses that $[T_e]$ is not thin.

- Compare it with the construction of \textit{nonhemimaximal degrees}. 
Subrequirements

\( \mathcal{R}_{e,i} \): There exist an interval \((x_{e,i}, z_{e,i})\) such that \([T_e]\) contains a branch extending \(\Phi_e(C) \upharpoonright x_{e,i}\), but not \(\Phi_e(C) \upharpoonright z_{e,i}\).

- If \(i\) is even, then all nodes in \(T_e\) extending \(\Phi_e(C) \upharpoonright x_{e,i}\), but not \(\Phi_e(C) \upharpoonright z_{e,i}\), will be put on \(S_e\).
- If \(i\) is odd, then all nodes in \(T_e\) extending \(\Phi_e(C) \upharpoonright x_{e,i}\), but not \(\Phi_e(C) \upharpoonright z_{e,i}\), will be terminated on \(S_e\).

So, if all the \(\mathcal{R}_{e,i}\)-subrequirements are satisfied, then \([S_e]\) is a subclass of \([T_e]\), containing, and also missing, infinitely many branches of \([T_e]\).
Subrequirements

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So, if all the \(\mathcal{R}_{e,i}\)-subrequirements are satisfied, then \([S_e]\) is a subclass of \([T_e]\), containing, and also missing, infinitely many branches of \([T_e]\).

- The crucial point is after one region is terminated, \(\Phi_e(C)\) will never come back to this region again, in the remainder of the construction.
Subrequirements

\( R_{e, i} \): There exist an interval \((x_{e, i}, z_{e, i})\) such that \([T_e]\) contains a branch extending \(\Phi_e(C) \upharpoonright x_{e, i}\), but not \(\Phi_e(C) \upharpoonright z_{e, i}\).

- If \(i\) is even, then all nodes in \(T_e\) extending \(\Phi_e(C) \upharpoonright x_{e, i}\), but not \(\Phi_e(C) \upharpoonright z_{e, i}\), will be put on \(S_e\).

- If \(i\) is odd, then all nodes in \(T_e\) extending \(\Phi_e(C) \upharpoonright x_{e, i}\), but not \(\Phi_e(C) \upharpoonright z_{e, i}\), will be terminated on \(S_e\).

So, if all the \(R_{e, i}\)-subrequirements are satisfied, then \([S_e]\) is a subclass of \([T_e]\), containing, and also missing, infinitely many branches of \([T_e]\).

- The crucial point is after one region is terminated, \(\Phi_e(C)\) will never come back to this region again, in the remainder of the construction.

- Threading strategy, for the consistency between \(R\)-strategies.
Upwards Density of NONTIN degrees

A joint work with Downey and Yang.
Upwards Density of NONTHIN degrees

A joint work with Downey and Yang.

- Fix $B$ as an incomplete c.e. set.

Construct a c.e. set $C$ such that the degree of $B \oplus C$ does not contain any member of thin classes.
Upwards Density of NONTINH degrees

A joint work with Downey and Yang.

- Fix $B$ as an incomplete c.e. set.

Construct a c.e. set $C$ such that the degree of $B \oplus C$ does not contain any member of thin classes.

 If $B \oplus C$ has NONTINH degree, then it is incomplete, as CDJS already proved that $0'$ contains elements of thin classes.
Requirements

$C$ is constructed to meet the following requirements:

$R_e$: if $\Phi_e(B \oplus C)$, $\psi_e(\Phi_e(B \oplus C))$ and $\Theta_e(\Phi_e(B \oplus C))$ are all total, then

- either $C \neq \psi_e(\Phi_e(B \oplus C))$, or
- $B \neq \Theta_e(\Phi_e(B \oplus C))$, or
- $\Phi_e(B \oplus C)$ is not in $[T_e]$, or
- $[T_e]$ is not thin.
Requirements

C is constructed to meet the following requirements:

\[ \mathcal{R}_e: \text{ if } \Phi_e(B \oplus C), \psi_e(\Phi_e(B \oplus C)) \text{ and } \Theta_e(\Phi_e(B \oplus C)) \text{ are all total, then } \]

- either \( C \neq \psi_e(\Phi_e(B \oplus C)) \), or
- \( B \neq \Theta_e(\Phi_e(B \oplus C)) \), or
- \( \Phi_e(B \oplus C) \text{ is not in } [T_e] \), or
- \( [T_e] \text{ is not thin.} \)

Concerns:
Requirements

$C$ is constructed to meet the following requirements:

$\mathcal{R}_e$: if $\Phi_e(B \oplus C)$, $\Psi_e(\Phi_e(B \oplus C))$ and $\Theta_e(\Phi_e(B \oplus C))$ are all total, then

- either $C \neq \Psi_e(\Phi_e(B \oplus C))$, or
- $B \neq \Theta_e(\Phi_e(B \oplus C))$, or
- $\Phi_e(B \oplus C)$ is not in $[T_e]$, or
- $[T_e]$ is not thin.

Concerns:

- Divergence
- Threading strategy

- $B$’s change can bring $\Phi_e(B \oplus C)$ into a terminated region. What shall we do?
Full density: True.
Full density: True.

Thanks!