Some applications of higher Demuth's theorem

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Demuth's theorem

Theorem (Demuth (1988))

If r_0 is Martin-Löf random and $z \leq_{tt} r_0$ is nonrecursive, then $z \equiv_T r_1$ for some Martin-Löf random real r_1.

The philosophy of Demuth’s theorem

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However, the appearance of the truth table reduction in the theorem makes the theorem a little imperfect.

Higher Demuth's theorem

Theorem (Chong and Y.)

If r_0 is Π^1_1-random and $z \leq_h r_0$ is nonhyperarithmetic, then $z \equiv_h r_1$ for some Π^1_1 random real r_1.

The partial relativization of the theorem can be read as:

If r_0 is $\Pi^1_1(x)$-random and $z \leq_h r_0$ is nonhyperarithmetic, then $z \equiv_h r_1$ for some $\Pi^1_1(x)$ random real r_1.

Sacks's theorem

Given a set of real A, let $\mathcal{U}_h(A) = \{y \mid \exists x \in A(y \geq_h x)\}$.

Theorem (Sacks (1969))

If x is no hyperarithmetic, then $\mathcal{U}_h(\{x\})$ is null.

Kripke's theorem

Sacks's theorem was greatly strengthened by Kripke.

Theorem (Kripke (1969))

If A is null, closed under hyperarithmetic equivalence relation and does not contain a hyperarithmetic real, then $\mathcal{U}_h(A)$ is null.

Proof.

Suppose not. Then fix a real x so that A does not contain any $\Pi^1_1(x)$-random real. But $\mathcal{U}_h(A)$ must contain such a real. Relativizing the higher Demuth's theorem to x, A must contain a $\Pi^1_1(x)$-random real, a contradiction. □

Antichains of hyperdegrees

An antichain of hyperdegrees is a set of hyperdegrees so that it has at least two elements and any two of them are incomparable.

Theorem (Y.)

- *If A has positive measure, then A contains two reals $x \leq_m y$ but $x \not\leq_h y$.*
- *There exists a maximal nonmeasurable antichain of hyperdegrees.*

A null maximal antichain of hyperdegrees.

Theorem (Chong and Y.)

There is a null maximal antichain A of hyperdegrees. Actually every Π^1_1 random real is strictly hyperarithmetically below some real in A.

Note that any nontrivial upper cone of hyperdegrees does not contain a maximal antichain.

The proof

1. If g is sufficiently generic, then g form a minimal pair (in the hyperdegree sense) with any Π^1_1-random reals;
2. For any hypdegree x, there are $2^{\aleph_0}$ many generic reals $\{g_\alpha\}_\alpha$ which mutually form an exact pair over the low cone of x.
3. By induction and try to avoid Π^1_1-random reals.

Some additional results.

Proposition

- *Given a set A of antichain of hyperdegrees. If $\mathcal{U}_h(A)$ is measurable, then it must be null.*
- *There is a nonmeasurable set A of hyperdegrees so that $\mathcal{U}_h(A)$ is conull.*

Measure theory of Turing degrees

Given a set of reals A, let $\mathcal{U}_T(A) = \{y \exists x \in A(y \geq_T x)\}$.

Theorem (Sacks (1963); de Leeuw, Moore, Shannon, and Shapiro (1956))

If x is not recursive, then $\mathcal{U}_T(\{x\})$ is null.

Theorem (Kurtz (1981) and Kautz (1991))

There is a null set A of Turing degrees which does not contain **0** *so that $\mathcal{U}_T(A)$ is conull.*

Antichains of Turing degrees

- There is nonmeasurable maximal antichain of Turing degrees.
- If A is antichain of Turing degrees so that $\mathcal{U}_T(A)$ is measurable, then so is A.

Jockusch's question

Question (Jockusch (2006))

- *Is there a measurable maximal antichain of Turing degrees?*
- *Is there a maximal antichain A of Turing degrees so that $\mathcal{U}_T(A)$ is null?*

The first question can be easily answered under *CH*.

Some classical genericity result (1)

Theorem (Kurtz and Kautz)

Every 2-random real is REA.

Theorem (Wang (2011))

If x is REA, then x is r.e. above some 1-generic real g.

Some classical genericity result (2)

Lemma (Chong and Y.)

If x is REA, then for any $n \geq 1$, there are n-many Turing incomparable 1-generic reals $\{g_i\}_{i \leq n}$ so that for any $i \neq j \leq n$, $g_i \oplus g_j \equiv_T x$.

The main theorem

Theorem (Chong and Y.)

There is a maximal antichain A of Turing degrees so that $\mu(\mathcal{U}_T(A)) = 1$.

The proof

1. Fix a null maximal antichain *B* of hyperdegrees so that each Π^1_1-random real is hyperarithmetically below some element in *B*;
2. Putting all the previous genericity results together and by an induction locally working below some element in *B*.

Additional results

Theorem (Chong and Y.)

- *There is a null maximal antichain A of Turing degrees so that $\mu(\mathcal{U}_T(A)) = 0$.*
- *There is a null maximal antichain A of Turing degrees so that $\mathcal{U}_T(A)$ is not measurable.*

Demuth's theorem on L-degrees

Theorem (Forklore)

If r_0 is random over L and $z \in L[r_0] \setminus L$, then $z \equiv_L r_1$ for some L-random real r_1.

So random forcing only adds random reals.

Kripke's results in L

Theorem

Suppose that for any real x, $\omega_1^{L[x]}$ is countable. The for any null set A of constructible degrees not containing $\mathbf{0}_L$, $\mathcal{U}_L(A)$ is null.

A question

Question

Is there a Π^0_1 set of maximal antichain of Turing degrees?

Thank you