Some applications of higher Demuth’s theorem

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June 16, 2014
Demuth’s theorem

Theorem (Demuth (1988))

If $r_0$ is Martin-Löf random and $z \leq_{tt} r_0$ is nonrecursive, then $z \equiv_T r_1$ for some Martin-Löf random real $r_1$. 
The philosophy of Demuth’s theorem

Demuth’s theorem is a kind of formalization of the following thesis

“Any information computed by a random oracle is either trivial or useless.”
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However, the appearance of the truth table reduction in the theorem makes the theorem a little imperfect.
Higher Demuth’s theorem

Theorem (Chong and Y.)

If $r_0$ is $\Pi^1_1$-random and $z \leq_h r_0$ is nonhyperarithmetic, then $z \equiv_h r_1$ for some $\Pi^1_1$ random real $r_1$.

The partial relativization of the theorem can be read as:

If $r_0$ is $\Pi^1_1(x)$-random and $z \leq_h r_0$ is nonhyperarithmetic, then $z \equiv_h r_1$ for some $\Pi^1_1(x)$ random real $r_1$. 
Sacks’s theorem

Given a set of real $A$, let $\mathcal{U}_h(A) = \{y \mid \exists x \in A(y \geq_h x)\}$.

**Theorem (Sacks (1969))**

If $x$ is no hyperarithmetic, then $\mathcal{U}_h(\{x\})$ is null.
Kripke’s theorem

Sacks’s theorem was greatly strengthened by Kripke.

**Theorem (Kripke (1969))**

*If A is null, closed under hyperarithmetic equivalence relation and does not contain a hyperarithmetic real, then $\mathcal{U}_h(A)$ is null.*

**Proof.**

Suppose not. Then fix a real $x$ so that $A$ does not contain any $\Pi^1_1(x)$-random real. But $\mathcal{U}_h(A)$ must contain such a real. Relativizing the higher Demuth’s theorem to $x$, $A$ must contain a $\Pi^1_1(x)$-random real, a contradiction.
An antichain of hyperdegrees is a set of hyperdegrees so that it has at least two elements and any two of them are incomparable.

**Theorem (Y.)**

- If $A$ has positive measure, then $A$ contains two reals $x \leq_m y$ but $x \not\leq_h y$.
- There exists a maximal nonmeasurable antichain of hyperdegrees.
A null maximal antichain of hyperdegrees.

**Theorem (Chong and Y.)**

There is a null maximal antichain $A$ of hyperdegrees. Actually, every $\Pi^1_1$ random real is strictly hyperarithmetically below some real in $A$.

Note that any nontrivial upper cone of hyperdegrees does not contain a maximal antichain.
The proof

1. If $g$ is sufficiently generic, then $g$ form a minimal pair (in the hyperdegree sense) with any $\Pi^1_1$-random reals;
2. For any hypdegree $x$, there are $2^{\aleph_0}$ many generic reals $\{g_\alpha\}_\alpha$ which mutually form an exact pair over the low cone of $x$.
3. By induction and try to avoid $\Pi^1_1$-random reals.
Some additional results.

**Proposition**

- Given a set $A$ of antichain of hyperdegrees. If $\mathcal{U}_h(A)$ is measurable, then it must be null.
- There is a nonmeasurable set $A$ of hyperdegrees so that $\mathcal{U}_h(A)$ is conull.
Measure theory of Turing degrees

Given a set of reals $A$, let $\mathcal{U}_T(A) = \{ y \exists x \in A (y \geq_T x) \}$.

Theorem (Sacks (1963); de Leeuw, Moore, Shannon, and Shapiro (1956))

If $x$ is not recursive, then $\mathcal{U}_T(\{ x \})$ is null.

Theorem (Kurtz (1981) and Kautz (1991))

There is a null set $A$ of Turing degrees which does not contain $0$ so that $\mathcal{U}_T(A)$ is conull.
Antichains of Turing degrees

- There is nonmeasurable maximal antichain of Turing degrees.
- If $A$ is antichain of Turing degrees so that $\mathcal{U}_T(A)$ is measurable, then so is $A$. 
Jockusch’s question

Question (Jockusch (2006))

- Is there a measurable maximal antichain of Turing degrees?
- Is there a maximal antichain $A$ of Turing degrees so that $\mathcal{U}_T(A)$ is null?

The first question can be easily answered under $CH$. 
Theorem (Kurtz and Kautz)
Every 2-random real is REA.

Theorem (Wang (2011))
If $x$ is REA, then $x$ is r.e. above some 1-generic real $g$. 
Lemma (Chong and Y.)

If $x$ is REA, then for any $n \geq 1$, there are $n$-many Turing incomparable 1-generic reals $\{g_i\}_{i \leq n}$ so that for any $i \neq j \leq n$, $g_i \oplus g_j \equiv_T x$. 
The main theorem

**Theorem (Chong and Y.)**

There is a maximal antichain $A$ of Turing degrees so that $\mu(\mathcal{U}_T(A)) = 1$. 
The proof

1. Fix a null maximal antichain $B$ of hyperdegrees so that each $\Pi^1_1$-random real is hyperarithmetically below some element in $B$;

2. Putting all the previous genericity results together and by an induction locally working below some element in $B$. 
Theorem (Chong and Y.)

- There is a null maximal antichain $A$ of Turing degrees so that $\mu(\mathcal{U}_T(A)) = 0$.
- There is a null maximal antichain $A$ of Turing degrees so that $\mathcal{U}_T(A)$ is not measurable.
Demuth’s theorem on $L$-degrees

**Theorem (Forklore)**

If $r_0$ is random over $L$ and $z \in L[r_0] \setminus L$, then $z \equiv_L r_1$ for some $L$-random real $r_1$.

So random forcing only adds random reals.
Kripke’s results in $L$

Theorem

Suppose that for any real $x$, $\omega_1^{L[x]}$ is countable. The for any null set $A$ of constructible degrees not containing $0_L$, $\mathcal{U}_L(A)$ is null.
A question

Question

Is there a $\Pi^0_1$ set of maximal antichain of Turing degrees?
Thank you