Nahm transform for parabolic integrable connections on the Riemann sphere

Szilárd Szabó

Budapest University of Technology and Rényi Institute of Mathematics
Budapest

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OUTLINE

Wild non-abelian Hodge theory on curves

Nahm transform

Hyper-Kähler isometry
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**Notations**

- $X$: smooth projective curve over $\mathbb{C}$
- $G = \text{Gl}_r(\mathbb{C})$
- $\mathbb{P}^1$: the Riemann sphere $\mathbb{C} \cup \{\infty\}$
- $P = \{p_0, p_1, \ldots, p_n\}$: a finite set of distinct points in $X$
- $\mathcal{O}$: sheaf of holomorphic functions
- $\Omega^k$: sheaf of smooth $k$-forms
- $\Omega^1$: sheaf of holomorphic 1-forms
Meromorphic connections and Higgs bundles on curves

Let $E$ be a holomorphic vector bundle of rank $r$ on $X$ and $D$ a meromorphic connection with singularities in $P$:

$$D : E \longrightarrow \Omega^1(*P) \otimes \mathcal{O}_E$$

satisfying the Leinbiz-rule.

Paralelly, let $\mathcal{E}$ be a holomorphic vector bundle of rank $r$ on $X$ and

$$\theta : \mathcal{E} \longrightarrow \Omega^1(*P) \otimes \mathcal{O}_E$$

a meromorphic Higgs field.
Fixing the irregular parts of $D$

We fix the behaviour of $D$ near the singular points as follows:

$$D = d + A_n \frac{dz}{z^n} + \cdots + A_2 \frac{dz}{z^2} + O(z^{-1})dz$$

with respect to some local analytic coordinate $z$ and some holomorphic trivialisation, where

$$A_2, \ldots, A_n$$

belong to some torus $t \subset gl_r(C)$. Let

$$H \subset Gl_r(C)$$

stand for the common centraliser of $A_2, \ldots, A_n$ and $\mathfrak{h}$ for its Lie-algebra.
Fixing the irregular parts of $\theta$

Parallely, we assume

$$\theta = T_n \frac{dz}{z^n} + \cdots + T_2 \frac{dz}{z^2} + O(z^{-1})dz$$

with respect to some trivialisation, where

$$T_k = \frac{A_k}{2} \quad (2 \leq k \leq n).$$
Parabolic structure at singular points

A compatible parabolic structure for $D$ is the choice of an element

$$\beta \in t_\mathbb{R}.$$ 

Up to conjugation we may assume $t$ consists of diagonal matrices, so we have

$$\beta = \text{diag}(\beta_1, \ldots, \beta_r).$$

Similarly, a compatible parabolic structure for $\theta$ is the choice of

$$\alpha = \text{diag}(\alpha_1, \ldots, \alpha_r) \in t_\mathbb{R}.$$

To $\alpha$ we associate the parabolic subgroup

$$P_\alpha = \{ g \in \text{Gl}_r(\mathbb{C}) | \ z^\alpha g z^{-\alpha} \text{ exists as } z \to 0 \}$$

and similarly we get $P_\beta$ with Lie-algebras $p_\alpha, p_\beta$ respectively.
Residues

We assume that

\[ A_1 \in \mathcal{O} \subset \mathfrak{h} \cap \mathfrak{p}_\beta \]

is in a fixed semi-simple adjoint orbit, defined by eigenvalues

\[ \mu_1, \ldots, \mu_r \]

and similarly,

\[ T_1 \in \mathcal{O}' \subset \mathfrak{h} \cap \mathfrak{p}_\alpha \]

is in a fixed semi-simple adjoint orbit, defined by eigenvalues

\[ \lambda_1, \ldots, \lambda_r. \]

These parameters are subject to Simpson’s relations

\[ \alpha_i = \Re(\mu_i), \quad \lambda_i = \frac{\mu_i - \beta_i}{2}. \]
**Stability of connections**

The parabolic degree and slope of $E$ are defined respectively as

$$\text{par-deg}(E) = \deg(E) + \sum_{j=0}^{n} \sum_{k=1}^{r} \beta^j_k$$

and

$$\text{par-slope}(E) = \frac{\text{par-deg}(E)}{\text{rank}(E)}.$$
Stability of connections

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$(E, D)$ is said to be parabolically stable if for all non-trivial proper subbundle $F \subset E$ such that $\text{Im}(D|_{F}) \subset \Omega^{1}(\ast P) \otimes F$, one has

$$\text{par-slope}(F) < \text{par-slope}(E).$$
**Stability of Higgs bundles**

The parabolic degree and slope of $\mathcal{E}$ are defined respectively as

$$\text{par-deg}(\mathcal{E}) = \deg(\mathcal{E}) + \sum_{j=0}^{n} \sum_{k=1}^{r} \alpha_{j}^{k}$$

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$(\mathcal{E}, \theta)$ is said to be parabolically stable if for all non-trivial proper subbundle $\mathcal{F} \subset \mathcal{E}$ such that $\text{Im}(\theta|_{\mathcal{F}}) \subset \Omega^{1}(\ast P) \otimes \mathcal{F}$, one has

$$\text{par-slope}(\mathcal{F}) < \text{par-slope}(\mathcal{E}).$$
Adapted Hermitian metrics

A Hermitian fiber metric $h$ is adapted to the parabolic structure of $(E, D)$ (respectively $(E, \theta)$) if near all $p_j \in X$ it is mutually bounded with

$$\text{diag}(|z_j|^2 \beta^j_k)_{k=1,\ldots,r},$$

(respectively $\text{diag}(|z_j|^2 \alpha^j_k)$) where $z_j$ is a local holomorphic coordinate of $X$ vanishing at $p_j$. 
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(respectively $\text{diag}( |z_j|^2\alpha^j_k )$) where $z_j$ is a local holomorphic coordinate of $X$ vanishing at $p_j$.

**Remark**

*Without the semi-simplicity assumption on the residues, the form of the matrices involves logarithmic terms corresponding to the weight filtration too.*
Let \((E, D)\) be a meromorphic connection endowed with a parabolic structure, and \(h\) an adapted Hermitian metric on it. Consider the decomposition

\[ D = D^+ \ + \ \Phi \]

of \(D\) into \(h\)-unitary and self-adjoint parts respectively.
Harmonic metrics

Let \((E, D)\) be a meromorphic connection endowed with a parabolic structure, and \(h\) an adapted Hermitian metric on it. Consider the decomposition

\[ D = D^+ + \Phi \]

of \(D\) into \(h\)-unitary and self-adjoint parts respectively. Decompose these parts further according to bidegree:

\[ \Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1} \]

\[ D^+ = \partial^+ + \bar{\partial}^+ \]

\[ \Phi = \theta + \theta^*. \]
Harmonic metrics

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\[ D^+ = \partial^+ + \bar{\partial}^+ \]

\[ \Phi = \theta + \theta^*. \]

Then, \(h\) is said to be harmonic if

\[ \bar{\partial}^+ \theta = 0. \]
**Hermitian–Einstein metrics**

Let \((\mathcal{E}, \theta)\) be a meromorphic Higgs field endowed with a parabolic structure, and \(h\) an adapted Hermitian metric on it. Let

\[
D_{\text{Chern}}
\]

denote the Chern connection associated with \(\bar{\partial}\mathcal{E}\) and \(h\) and \(\theta^*\) the adjoint of \(\theta\) with respect to \(h\). Then, \(h\) is said to be Hermitian–Einstein if the connection

\[
D = D_{\text{Chern}} + \theta + \theta^*
\]

is flat. If this holds then \((D, h)\) solve Hitchin’s equations

\[
F_{D^+} + [\theta, \theta^*] = 0
\]

\[
\bar{\partial}^+ \theta = 0.
\]
**Wild non-abelian Hodge theory**

**Theorem (O. Biquard – P. Boalch 2004)**

1. Let \((E, D)\) be a parabolically stable meromorphic integrable connection of parabolic degree 0, with polar parts fixed as above. Then, there exists a unique adapted harmonic metric \(h\) (up to a constant).

2. Let \((\mathcal{E}, \theta)\) be a parabolically stable meromorphic Higgs bundle of parabolic degree 0, with polar parts fixed as above. Then, there exists a unique adapted Hermitian–Einstein metric \(h\) (up to a constant).

3. For generic values of the parameters the moduli space \(\mathcal{M}^{\text{irr}}\) of irreducible solutions of Hitchin’s equations with prescribed singularity data up to unitary gauge transformations is a smooth complete hyper-Kähler manifold.
From now on, the parameters are assumed to be generic so that $\mathcal{M}$ is smooth and complete, and $(E, D) \in \mathcal{M}$ with harmonic Hermitian metric $h$.

The tangent space of $\mathcal{M}$ at $(E, D)$ is given by

$$T_{(E,D)}\mathcal{M} = \{ a \in L^2(X, \Omega^1_X \otimes \text{End}(E)) : D(a) = 0, D^*(a) = 0 \}.$$ 

The Atiyah–Bott Riemannian structure is given by the natural $L^2$-metric

$$\sqrt{-1} \int_X \text{tr}(a \wedge a^*).$$
The de Rham and Dolbeault complex structures are respectively given by

\[ J(a) = \sqrt{-1}a, \quad I(a) = \sqrt{-1}(a^{0,1})^* - \sqrt{-1}(a^{1,0})^*. \]

Write

\[ (D + a)^+ = (\partial^+ - \dot{A}^*) + (\bar{\partial}^+ + \dot{A}) \]

with \( \dot{A} \) of type \((0, 1)\) and let

\[ \phi + \dot{\phi} + \dot{\phi}^*, \quad \dot{\phi} \in \Omega^{1,0} \]

denote the self-adjoint part of \( D + a \). Then we have

\[ I(\dot{A}, \dot{\phi}) = (\sqrt{-1} \dot{A}, \sqrt{-1} \dot{\phi}). \]
Dolbeault holomorphic symplectic structure

Given a hyper-Kähler manifold $(M, g, I, J, K)$, let

$$\omega_J(., .) = g(., J.), \quad \omega_K(., .) = g(., K.)$$

be the Kähler forms and

$$\Omega_I = \omega_J + \sqrt{-1}\omega_K.$$ 

Then $(I, \Omega_I)$ defines a holomorphic symplectic structure on $M$. For $M$ with $g, I, J, K$ defined as above this structure is given by

$$\Omega_I(((\dot{A}, \dot{\Phi}), (\dot{B}, \dot{\Psi})) = \int_X \text{tr}(\dot{\Psi} \wedge \dot{A} - \dot{\Phi} \wedge \dot{B}).$$
Isometries between moduli spaces

Question

Are there isometries between the wild Hitchin moduli spaces?
Isometries between moduli spaces

**Question**

Are there isometries between the wild Hitchin moduli spaces?

Yes, some are given by Nahm transformation.
Assumption on points at finite distance

From now on we let $X = \mathbb{P}^1$, $p_1, \ldots, p_n \in \mathbb{C}$, $p_0 = \infty$. $D$ is supposed to have a logarithmic singularity (i.e., $n = 1$) at $p_j$ for $j \in \{1, \ldots, n\}$: in a local trivialisation of $E$ near $p_j$, one has

$$D = d + \frac{A^j(z)}{z - p_j} dz,$$

where $A^j$ is a holomorphic matrix-valued function defined near $p_j$. Furthermore, the residue

$$A^j(p_j) = \text{diag}(0, \ldots, 0, \mu^{j}_{r_j+1}, \ldots, \mu^{j}_r),$$

is diagonal, with $\mu^{j}_k$ non-zero and generic.
Assumption at infinity

$D$ is supposed to have an irregular singularity with $n - 1 = 1$ at infinity: in a local trivialisation of $E$ near $\infty$, one has

$$D = d + Adz + B \frac{dz}{z} + \text{lower order terms},$$

where

$$A = \text{diag}(\xi_1, \ldots, \xi_1, \ldots \ldots \ldots, \xi_{n'}, \ldots, \xi_{n'})$$
$$B = \text{diag}(\mu_1^0, \ldots, \mu_{a_2}^0, \ldots \ldots, \mu_{1+a_n'}^0, \ldots, \mu_{r}^0)$$

(the leading order term and residue, respectively). Here the $\xi_k$ are pairwise distinct constants, and the $\mu_i^0$ are generic non-zero.

(Notation: $a_1 = 0, a_{n'+1} = r$.)
Let \( \hat{C} \) and \( \hat{P}^1 \) be another copy of \( C \) and \( P^1 \) respectively. Call \( \hat{P} = \{\xi_1, \ldots, \xi_{n'}\} \) the transformed singular set. For any \( \xi \in \hat{C} \setminus \hat{P} \), define the twisted connection as

\[
D_\xi = D - \xi dz.
\]
Expontential twist

Let $\hat{\mathcal{C}}$ and $\hat{\mathbb{P}}^1$ be another copy of $\mathcal{C}$ and $\mathbb{P}^1$ respectively. Call $\hat{\mathbb{P}} = \{\xi_1, \ldots, \xi_{n'}\}$ the transformed singular set. For any $\xi \in \hat{\mathcal{C}} \setminus \hat{\mathbb{P}}$, define the twisted connection as

$$D_\xi = D - \xi dz.$$ 

Let $D_\xi^*$ stand for the adjoint operator of $D_\xi$ with respect to $h$, and define the twisted Laplace operator

$$\Delta_\xi = D_\xi D_\xi^* + D_\xi^* D_\xi$$

as an unbounded operator acting on $L^2(\Omega^1 \otimes E)$. 
The kernel of the twisted Laplacian

For any $\xi \in \widehat{C \setminus P}$, the twisted Laplace operator

$$\Delta_\xi : L^2(\Omega^1 \otimes E) \longrightarrow L^2(\Omega^1 \otimes E)$$

has finite dimensional kernel.
The kernel of the twisted Laplacian

For any $\xi \in \hat{C} \setminus \hat{P}$, the twisted Laplace operator

$$\Delta_\xi : L^2(\Omega^1 \otimes E) \longrightarrow L^2(\Omega^1 \otimes E)$$

has finite dimensional kernel. The vector spaces $\text{ker}(\Delta_\xi)$ form a smooth family of finite-dimensional subspaces of $L^2(\Omega^1 \otimes E)$ of the same dimension, parametrized by $\hat{C} \setminus \hat{P}$. 
**Definition**

The smooth vector bundle with fiber over $\xi \in \hat{\mathcal{C}} \setminus \hat{\mathcal{P}}$ equal to $\ker(\Delta_\xi)$ is called the transformed smooth vector bundle. We denote it by $\hat{E}$, and its fiber over $\xi$ by $\hat{E}_\xi$. 
**Definition**

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Let $\varphi(z), \psi(z) \in \hat{E}_\xi$ for some $\xi \in \mathcal{C} \setminus \mathcal{P}$.

**Definition**

The transformed Hermitian metric $\hat{h}$ is defined on the fiber $\hat{E}_\xi$ by the formula

$$\hat{h}(\varphi, \psi) = \int_{\mathcal{C}} h(\varphi(z), \psi(z)).$$
THE TRANSFORMED FLAT CONNECTION

$L^2$-metric on sections of $\Omega^1 \otimes E$ induces an orthogonal projection

$$\pi_\xi : L^2(P^1, \Omega^1 \otimes E) \rightarrow \hat{E}_\xi.$$ 

Fix $\xi_0 \in \hat{C} \setminus \hat{P}$ and let $\varphi_1(z), \ldots, \varphi_{r'}(z)$ be a basis of $\hat{E}_{\xi_0}$. These sections are exponentially decreasing at infinity. In particular, for all $\xi$ sufficiently close to $\xi_0$ in $\hat{C} \setminus \hat{P}$ one can consider the sections

$$\varphi_j(\xi; z) = \pi_\xi(e^{(\xi-\xi_0)z} \varphi_j(z)) \in \hat{E}_\xi.$$
The transformed flat connection

$L^2$-metric on sections of $\Omega^1 \otimes E$ induces an orthogonal projection

$$\pi_\xi : L^2(\mathbb{P}^1, \Omega^1 \otimes E) \longrightarrow \hat{E}_\xi.$$

Fix $\xi_0 \in \hat{C} \setminus \hat{P}$ and let $\varphi_1(z), \ldots, \varphi_{r'}(z)$ be a basis of $\hat{E}_{\xi_0}$. These sections are exponentially decreasing at infinity. In particular, for all $\xi$ sufficiently close to $\xi_0$ in $\hat{C} \setminus \hat{P}$ one can consider the sections

$$\varphi_j(\xi; z) = \pi_\xi(e^{(\xi-\xi_0)z}\varphi_j(z)) \in \hat{E}_\xi.$$

**Definition**

The transformed flat connection $\hat{D}$ on $\hat{E}$ is defined by the basis of local parallel sections $\varphi_j(\xi; z)$ for $j \in \{1, \ldots, r'\}$. 
**Definition**

The metric extension of $\hat{E}$ over $\xi_l \in \hat{P}$ (respectively $\hat{\infty}$) is the lattice consisting of local holomorphic sections outside of $\xi_l$ (respectively $\hat{\infty}$) whose $\hat{h}$-norm is bounded from above by a constant.

We denote

$$(\hat{E}, \hat{D}, \hat{h}) = \mathcal{N}(E, D, h).$$
Properties of the transform

Theorem (Sz 2008)

- $\hat{D}$ is an integrable connection on $\hat{E}$, with logarithmic singularities in $\xi_1 \in \hat{P}$ and an irregular singularity of Poincaré-rank 1 ($n = 2$) at $\hat{\infty}$.

- The metric extension induces a parabolic structure on $\hat{E}$ at the singular points.

- The corresponding eigenvalues and parabolic weights transform according to the diagrams on the next two slides. In particular, $\hat{E}$ is of rank $\sum_{j=1}^{n} (r - r_j)$ and of parabolic degree 0.

- The metric $\hat{h}$ is harmonic for $\hat{D}$.

- Nahm transform $\mathcal{N}$ is involutive (up to a sign).
## Transform of the eigenvalues

\[\begin{array}{cccccc}
\infty & p_1 & \cdots & p_n \\
- & - & - & - & - & - \\
\xi_1 + z^{-1} \mu_1^0 & 0 & 0 \\
\vdots & 0 & \vdots \\
\xi_1 + z^{-1} \mu_{a_2}^0 & : & 0 \\
\vdots & 0 & \mu_{r_{n+1}}^n \\
\xi_{n'} + z^{-1} \mu_{1+a_{n'}}^0 & \mu_{r_1+1}^1 & \vdots \\
\vdots & \vdots & \vdots \\
\xi_{n'} + z^{-1} \mu_{r}^0 & \mu_r^1 & \mu_r^n \\
\end{array}\]
Transform of the eigenvalues

\[\begin{align*}
\infty &\quad p_1 &\cdots &\quad p_n &\quad \hat{\infty} \\
- & - & - & - & - \\
\xi_1 + z^{-1} \mu_1^0 & 0 & 0 & -p_1 + \zeta^{-1} \mu_{r_1+1}^1 \\
\vdots & 0 & \vdots & \vdots & \vdots \\
\xi_1 + z^{-1} \mu_{a_2}^0 & \vdots & 0 & -p_1 + \zeta^{-1} \mu_{r}^1 \\
\vdots & \mu_{r_1+1}^1 & \mu_{r_n+1}^n & \vdots & \vdots \\
\xi_{n'} + z^{-1} \mu_{1+a_{n'}}^0 & \mu_{r_1+1}^1 & \vdots & -p_n + \zeta^{-1} \mu_{r_{n+1}}^n \\
\vdots & \mu_{r}^1 & \mu_{r}^n & \mu_{r}^n & -p_n + \zeta^{-1} \mu_{r}^n \\
\xi_{n'} + z^{-1} \mu_{r}^0 & \mu_{r}^1 & \mu_{r}^n & \mu_{r}^n & \mu_{r}^n
\end{align*}\]
Transform of the eigenvalues

\[
\begin{align*}
\infty & \quad p_1 & \quad \cdots & \quad p_n & \quad \infty & \quad \xi_1 \\
- & - & - & - & - & - \\
\xi_1 + z^{-1} \mu_1^0 & \quad 0 & \quad 0 & \quad -p_1 + \zeta^{-1} \mu_{r_1+1} & \quad 0 \\
\vdots & \quad 0 & \quad \vdots & \quad \vdots & \quad 0 \\
\xi_1 + z^{-1} \mu_{a_2}^0 & \quad \vdots & \quad 0 & \quad -p_1 + \zeta^{-1} \mu_1 & \quad \vdots \\
\vdots & \quad 0 & \quad \vdots & \quad \mu_{r_{n+1}} & \quad 0 \\
\xi_{n'} + z^{-1} \mu_{1+a_{n'}}^0 & \quad \mu_{r_1+1}^1 & \quad \vdots & \quad -p_n + \zeta^{-1} \mu_{r_{n+1}} & \quad \mu_{a_2}^0 \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
\xi_{n'} + z^{-1} \mu_{r}^0 & \quad \mu_r & \quad \mu_r^n & \quad -p_n + \zeta^{-1} \mu_r & \quad \mu_{a_2}^0
\end{align*}
\]
## Transform of the Eigenvalues

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<th>$\infty$</th>
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<th>$\xi_1$</th>
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## Transform of the Weights

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Nahm transform

Szilárd Szabó, Budapest
# Transform of the weights

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Nahm transform

Szilárd Szabó, Budapest
Theorem (Sz 2014)

Nahm transform is a hyper-Kähler isometry.

Strategy of proof: show

\[ I \mapsto \hat{I} \]
\[ J \mapsto \hat{J} \]
\[ \Omega_I \mapsto \Omega_{\hat{I}} \]
The transformation of the complex structure $I$ follows from an algebraic interpretation

$$L^2 H^1(D_\xi) \cong H^1(\mathcal{E} \xrightarrow{\theta-\xi dz} \mathcal{F})$$

as the hypercohomology of the Dolbeault complex (Aker–Sz 2014). The transformation of the complex structure $J$ follows from identification with minimal extension followed by Fourier–Laplace transform of the underlying holonomic $\mathcal{D}$-module (Sz 2012). From now on we will focus on the transformation of $\Omega_I$. 
Beauville–Narasimhan–Ramanan correspondence

Set

\[ L = \Omega^1_{P1}(P) = \Omega^1_{P1}(p_1 + \cdots + p_n + 2 \cdot \infty) \]

and consider the ruled surface

\[ Z = P(\mathcal{O}_{P1} \oplus L) \xrightarrow{\pi} P^1 \]

with relatively ample line bundle \( \mathcal{O}(1) \) and global sections

\[ x \in H^0(Z, \mathcal{O}(1) \otimes \pi^* L), \quad y \in H^0(Z, \pi^* L). \]

Consider the cokernel sheaf \( M(\mathcal{E}, \theta) \) defined by

\[ 0 \rightarrow \pi^* (\mathcal{E} \otimes L^\vee) \xrightarrow{\pi^* \theta \otimes y - \pi^* Id_{\mathcal{E}} \otimes x} \pi^* \mathcal{E} \otimes \mathcal{O}(1) \rightarrow M(\mathcal{E}, \theta) \rightarrow 0. \]
Beauville–Narasimhan–Ramanan Correspondence, cont’d

It is possible to recover \((\mathcal{E}, \theta)\) from \(M(\mathcal{E}, \theta)\):

\[
\mathcal{E} = \pi_* M(\mathcal{E}, \theta), \quad \theta = \pi_*(x : M \to M \otimes \pi_* L \otimes \mathcal{O}(1)).
\]

The support \(S(\mathcal{E}, \theta)\) of \(M(\mathcal{E}, \theta)\) is called spectral curve. The above associations induce an equivalence between the categories of

- Higgs bundles with integral spectral curve \(S\)

and

- torsion sheaves of pure dimension 1 and of rank 1 with irreducible support away from \((y)\)
Hilbert scheme of curves

Notice: $Z$ is a holomorphic Poisson surface with Liouville symplectic form $\omega$ degenerating along

$$D_\infty = \pi^{-1} P + 2 \cdot (y).$$

Given $r$ consider the Hilbert scheme

$$\text{Hilb}(r)$$

of curves $S \subset Z$ having the same Hilbert polynomial as a generic $r$ to 1 cover of $\mathbb{P}^1$ in $Z$,

$$\text{Hilb}^0(r) \subset \text{Hilb}(r)$$

the connected component of a given $S_0$, and

$$B \subset \text{Hilb}^0(r)$$

the Zariski open subset parameterising smooth irreducible curves $S$ not contained in $D_\infty$. 

Moduli spaces of sheaves on Poisson surfaces

Consider moreover the relative Picard bundle

$$\text{Pic}^d(Z) \to B$$

whose fiber over $b \in B$ is the set of isomorphism classes of degree $d$ line bundles over $S_b$.

**Theorem (Donagi, Markman 1996)**

$B$ is smooth and $\text{Pic}^d(Z)$ has a canonical Poisson structure whose symplectic leaves are obtained by prescribing the intersection of the curves $S$ with $D_\infty$. 

Wild non-abelian Hodge theory on curves

Nahm transform

Hyper-Kähler isometry
Deformation theory of sheaves

The deformation theory of Pic$^d(Z)$ at a given sheaf $M$ is given by the global Ext-groups

$$\text{Ext}^*_\mathcal{O}_Z (M, M)$$

and the restriction of the Poisson structure $\Omega_{\text{Mukai}}$ to the symplectic leaves is induced by the Yoneda product

$$\cup : \text{Ext}^1_{\mathcal{O}_Z} (M, M) \times \text{Ext}^1_{\mathcal{O}_Z} (M, M) \rightarrow \text{Ext}^2_{\mathcal{O}_Z} (M, M)$$

followed by Serre duality.
Matching the symplectic structures

Consider two 1-parameter families

\[(\mathcal{E}(t), \Phi(t)), \quad (\mathcal{E}(x), \Phi(x))\]

of elements of \(\mathcal{M}\) for \(t, x \in \mathbb{C}\) both specialising to \((\mathcal{E}, \Phi) \in \mathcal{M}\) at \(t = 0\) and \(x = 0\) respectively. They give rise to

\[T, X \in T_{(\mathcal{E}, \Phi)}\mathcal{M}.\]

The associated families of spectral sheaves

\[\mathcal{M}_{(\mathcal{E}(t), \Phi(t))}, \quad \mathcal{M}_{(\mathcal{E}(x), \Phi(x))}\]

in \(\text{Pic}^d(Z)\) then give rise to tangent vectors

\[\tilde{T}, \tilde{X} \in T_{\mathcal{M}_{(\mathcal{E}, \Phi)}} \text{Pic}^d(Z).\]
Then we have the

**Key Formula**

\[ \Omega_I(T, X) = \Omega_{\text{Mukai}}(\tilde{T}, \tilde{X}). \]

Proven in particular cases by Hurtubise (1996) and Hurtubise–Harnad (2008).
END OF THE PROOF USING KEY FORMULA

Known:

\[(\mathcal{E}, \theta) \quad (\hat{\mathcal{E}}, \hat{\theta})\]

have isomorphic spectral sheaves

\[M(\mathcal{E}(t), \Phi(t)) \cong M(\hat{\mathcal{E}}(t), \hat{\Phi}(t))\]

on the open surface

\[T^*(\mathbf{C} \setminus P).\]

Therefore, the Key Formula applied to the vectors

\[\hat{T} = T_{(\mathcal{E}, \Phi)} \mathcal{N}(T), \quad \hat{X} = T_{(\mathcal{E}, \Phi)} \mathcal{N}(X)\]

shows that

\[\Omega_{\hat{\imath}}(\hat{T}, \hat{X}) = \Omega_{\text{Mukai}}(\tilde{T}, \tilde{X})\]

too.
Computing global Ext groups

By definition, the Ext-groups $\text{Ext}^*_\mathcal{O}_Z(M, M)$ are computed by the hypercohomology of the following complex of coherent sheaves on $Z$:

$$\mathcal{H}om_{\mathcal{O}_Z}(\pi^*\mathcal{E} \otimes \mathcal{O}(1), M(\mathcal{E}, \Phi)) \longrightarrow \mathcal{H}om_{\mathcal{O}_Z}(\pi^*(\mathcal{E} \otimes_{\mathcal{O}_C} L^\vee), M(\mathcal{E}, \Phi))$$

in degrees 0, 1 where the arrow is given by

$$\mathcal{H}om(\pi^*\Phi \otimes y - \pi^*\text{Id}_\mathcal{E} \otimes x, \text{Id}_M).$$

The sheaves of this complex are supported on $S(\mathcal{E}, \Phi)$ and its push-forward by $\pi$ is

$$\mathcal{E}nd_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{E}) \xrightarrow{\text{ad}_\Phi} \mathcal{E}nd_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{E}) \otimes_{\mathcal{O}_{\mathbb{P}^1}} L.$$
Let $\Delta$ be a small analytic disc in $\mathbb{P}^1 \setminus P_{\text{red}}$ such that

$$S \cap \pi^{-1}(\Delta) = S_1 \cup \cdots \cup S_r$$

with

$$\pi_j = \pi|_{S_j} : S_j \to \Delta$$

bianalytic. Let furthermore $z \in \Delta$ be a local holomorphic coordinate and

$$x_1(z), \ldots, x_r(z)$$

the eigenvalues of $\theta$ over $\Delta$ so that

$$S_j = \text{Im}(x_j).$$

Then

$$M|_{S_j}$$

is a holomorphic line bundle over $S_j$ with some holomorphic trivialisation $m_j$. 
**Diagonalisation**

Set

\[ \mathbf{e}_j = \pi_\ast \mathbf{m}_j; \]

with respect to the frame \( \mathbf{e}_1, \ldots, \mathbf{e}_r \) one has

\[ \theta(z) = \text{diag}(x_1(z), \ldots, x_r(z)) \]

and

\[ \mathbf{m}_j = [\mathbf{e}_j] \in \text{coker}(\theta - x_j) \]

on \( S_j \). A frame for

\[ \mathcal{H}om_{\mathcal{O}_Z}(\pi^* \mathcal{E} \otimes \mathcal{O}(1), M(\mathcal{E}, \Phi))|_{S_j} \]

is then given by

\[ \pi^* \mathbf{e}_1^\vee \otimes \mathbf{m}_j, \ldots, \pi^* \mathbf{e}_r^\vee \otimes \mathbf{m}_j. \]
Let

\[ w_j = \pi_j^{-1}(z) \]

be the local holomorphic coordinate on \( S_j \), then a frame for

\[ \mathcal{H}om_{\mathcal{O}_Z}(\pi^*(\mathcal{E} \otimes_{\mathcal{O}_{P^1}} L^\vee), M(\mathcal{E}, \Phi))|_{S_j} \]

is given by

\[ \pi^*e_1^\vee \otimes m_j dw_j, \ldots, \pi^*e_r^\vee \otimes m_j dw_j. \]
Dolbeault representatives of tangent vectors

Let

\[ T = [(\dot{A}, \dot{\Phi})], \quad X = [(\dot{B}, \dot{\Psi})] \in T_{(\mathcal{E}, \theta)}\mathcal{M} \cong H^1(X, \text{ad}_\theta) \]

with

\[ \dot{A} = a d\bar{z}, \quad \dot{\Phi} = \phi dz \]
\[ \dot{B} = b d\bar{z}, \quad \dot{\Psi} = \psi dz \]

where

\[ a = (a_{ij}), \ b, \ \phi, \ \psi : \Delta \to \mathfrak{gl}_r(\mathbb{C}) \]

are \( L^2 \) matrices of endomorphisms of \( E \) with respect to the framing \( e_1, \ldots, e_r \) satisfying

\[ \bar{\partial}(\phi dz) + [ad\bar{z}, \theta] = 0, \quad \bar{\partial}(\psi dz) + [bd\bar{z}, \theta] = 0. \]
LIFTING TANGENT VECTORS

Define now

\[(\tilde{A}, \tilde{\Phi}), \quad (\tilde{B}, \tilde{\Psi})\]
on \(S_j\) as \((\tilde{a}_j d \tilde{w}_j, \tilde{\phi}_j d w_j)\) and \((\tilde{b}_j d \tilde{w}_j, \tilde{\psi}_j d w_j)\) respectively, where

\[
\tilde{a}_j(w_j) : \pi^*_j e_i \mapsto a_{ij}(\pi(w_j)) m_j
\]
\[
\tilde{\phi}_j(w_j) : \pi^*_j e_i \mapsto \phi_{jj}(\pi(w_j)) m_j
\]

and

\[
\tilde{b}_j(w_j) : \pi^*_j e_i \mapsto b_{ij}(\pi(w_j)) m_j
\]
\[
\tilde{\psi}_j(w_j) : \pi^*_j e_i \mapsto \psi_{ij}(\pi(w_j)) m_j.
\]
The above local definitions then match up to define global $L^2$ sections

$$(\tilde{A}, \tilde{\Phi}), \quad (\tilde{B}, \tilde{\Psi})$$

away from $\pi^{-1}(P_{\text{red}} \cup R)$ where $R$ is the branch locus of $\pi : S \to C$.

The push-forwards of these sections by $\pi$ are equal to $(\dot{A}, \dot{\Phi})$ and $(\dot{B}, \dot{\Psi})$, respectively.

Finally, they are 1-cocycles in the Dolbeault resolution of

$$\mathcal{H}om_{\mathcal{O}_Z}(\pi^* \mathcal{E} \otimes \mathcal{O}(1), M(\mathcal{E}, \Phi)) \longrightarrow \mathcal{H}om_{\mathcal{O}_Z}(\pi^*(\mathcal{E} \otimes_{\mathcal{O}_C} L^\vee), M(\mathcal{E}, \Phi)).$$

In particular, they define elements

$$\tilde{T}, \tilde{X} \in \text{Ext}^1_{\mathcal{O}_Z}(M(\mathcal{E}, \theta), M(\mathcal{E}, \theta)).$$
**Identifying $\text{Ext}^2$**

A standard spectral sequence argument yields

$$\text{Ext}^2_{\mathcal{O}_Z}(M(\mathcal{E},\Phi), M(\mathcal{E},\Phi)) \cong H^1(Z, \text{Ext}^1_{\mathcal{O}_Z}(M(\mathcal{E},\Phi), M(\mathcal{E},\Phi))).$$

The Poisson bivector induces an isomorphism

$$\text{Ext}^1_{\mathcal{O}_Z}(M(\mathcal{E},\Phi), M(\mathcal{E},\Phi)) \cong K_S(-(S \cap \pi^{-1}(P))).$$

where $S$ stands for $S_{(\mathcal{E},\Phi)}$. We infer

$$\text{Ext}^2_{\mathcal{O}_Z}(M(\mathcal{E},\Phi), M(\mathcal{E},\Phi)) \cong H^1(S, K_S(-(S \cap \pi^{-1}(P))))$$

$$\cong H^0(S, \mathcal{O}_S(S \cap \pi^{-1}(P)))^\vee$$

$$\cong H^0(S, \mathcal{O}_S)^\vee \bigoplus_{i=1}^n \bigoplus_{k=1}^r \mathbb{C}^{m_k+1}_{(z_k,x_i)}.$$
**Yoneda product**

The Yoneda product $\tilde{T} \cup \tilde{X}$ of the lifted tangent vectors is given by composition of homomorphisms coupled with wedge product of differential forms. Therefore in local coordinates on the sheet $S_i$ it can be represented by

$$\sum_j (a_{ij} \psi_{ji} - b_{ij} \phi_{ji}) dw_i \wedge d\bar{w}_i \in \Omega^2(S_i).$$

The evaluation of this Yoneda product on the generator of $H^0(S, \mathcal{O}_S)$ is

$$\sum_{i=1}^{r} \int_{S_i} (a_{ij} \psi_{ji} - b_{ij} \phi_{ji}) dw_i \wedge d\bar{w}_i.$$
**Mukai form**

The changes of variables $w_i \simar z$ transform the above sum of integrals into

$$\int_{\Delta} \text{tr}(\dot{\Psi} \wedge \dot{A} - \dot{\Phi} \wedge \dot{B})$$

which is the expression computing $\Omega_I$ on $\Delta$.

The Mukai form evaluated on $\tilde{T}, \tilde{X}$ can then be obtained by globalising the above analysis, using a partition of unity argument.

To make sure the formulae converge at $P_{\text{red}}$ one uses the estimate

$$|a_{ij} \psi_{ji}| \leq Cr^{-2+2\delta}$$

of Biquard and Boalch for some $\delta > 0$ as $r = |z - p_k| \to 0$. 