Nonlinear Elliptic Equations with Fractional Laplacian

Jianfu Yang

(Dept. of Math., Jiangxi Normal University)
Lévy process and pseudo differential operators
Let $(\Omega, \mathcal{F}, P)$ be a probability space, then measurable mappings from $\Omega$ to $\mathbb{R}^N$ are called random variables.

A stochastic process is a family of random variables $X = (X(t), t \geq 0)$ that all defined on the same probability space.

Let $X$ be a random variable taking values in $\mathbb{R}^N$ with law $\mu_X$. We say $X$ is infinitely divisible if, for all $n \in \mathbb{N}$, there exist i.i.d. random variables $Y_1^{(n)}, \ldots, Y_n^{(n)}$ such that

$$X = Y_1^{(n)} + \cdots + Y_n^{(n)}.$$
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Lévy process

- Let $\Phi_X(u) = \mathbb{E}(e^{i(u,X)})$ denote the characteristic function of $X$, where $u \in \mathbb{R}^N$, $E(f(X)) := \int_{\Omega} f(X(\omega))P(d\omega)$ is the expectation for $f(X)$.

- Let $\mathcal{M}(\mathbb{R}^N)$ denote the set of all Borel probability measures on $\mathbb{R}^N$. If $\mu \in \mathcal{M}(\mathbb{R}^N)$, then

  $$\Phi_\mu(u) = \int_{\mathbb{R}^N} e^{i(u,y)}\mu(dy).$$

- Let $\nu$ be a Borel measure defined on $\mathbb{R}^N \setminus \{0\}$. $\nu$ is a Lévy measure if

  $$\int_{\mathbb{R}^N \setminus \{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty.$$
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Lévy-Khintchine formula $\mu \in \mathcal{M}(\mathbb{R}^N)$ is infinitely divisible if there exists a vector $b \in \mathbb{R}^N$, a positive definite symmetric $N \times N$ matrix $A$ and a Lévy measure $\nu$ on $\mathbb{R}^N \setminus \{0\}$ such that, for all $u \in \mathbb{R}^N$,

$$
\Phi_\mu(u) = \exp\{i(b, u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^N \setminus \{0\}} \left[ e^{i(u, y)} - 1 - i(u, y)\chi_{B_1(0)}(y) \right] \nu(dy) \}.\tag{1}
$$

Conversely, any mapping of the form (1) is the characteristic function of an infinitely divisible probability measure on $\mathbb{R}^N$. 
The members of the triple \((b, A, \nu)\) are called the characteristics of the infinitely divisible random variable \(X\).

Examples: Gaussian random variable \(X\): \(b\) is the mean, \(A\) is the covariance matrix, \(\mu = 0\):

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Poisson random variable \(X\): \(b = 0, A = 0, \nu = c\delta_1\):

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Compound Poisson random variable $X$:

\[ b = c \int_{B_1} \chi \mu(dy), A = 0, \nu = c \mu, c > 0, \text{ i.e.} \]

\[ \Phi_X(u) = \exp\left\{ \int_{\mathbb{R}^N} (e^{i(u,y)} - 1) c \mu Z(dy) \right\}, \]

where $X = Z(1) + \cdots + Z(n)$.

We write the characteristic function

\[ \Phi_{\mu}(u) = e^{\eta(u)}. \]

\[ \eta : \mathbb{R}^N \to \mathbb{C} \] is called Lévy symbol, which is the symbol of a pseudo-differential operator.
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Let $X = (X(t), t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$.

- We say $X$ has **independent increments** if for each $n \in \mathbb{N}$ and each $0 \leq t_1 \leq \cdots \leq t_{n+1} < \infty$ the random variables $(X(t_{j+1}) - X(t_j), 1 \leq j \leq n)$ are independent;
- it has **stationary increments** if each

$$X(t_{j+1}) - X(t_j) = X(t_{j+1} - t_j) - X(0).$$
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- \( X \) is a Lévy process if:
  - \( X(0) = 0 \) a.s;
  - \( X \) has independent and stationary increments;
  - \( X \) is stochastically continuous, i.e. for all \( a > 0 \) and all \( s \geq 0 \)
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- If $X$ is a Lévy process, then $X(t)$ is infinitely divisible for each $t \geq 0$.
- If $X$ is a Lévy process, then
  \[ \Phi_X(t)(u) = e^{t\eta(u)} \]
  for each $u \in \mathbb{R}^N, t \geq 0$, where $\eta$ is the Lévy symbol of $X(1)$.
- Let $(b, A, \nu)$ be the characteristics of $X(1)$. By Lévy-Khinchine formula
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  \mathbb{E}(e^{i(u,X(t))}) = \exp\{t[i(b, u) - \frac{1}{2}(u, Au)]
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Pseudo-differential operators
Let $X = \{X(t)\}$ be a Lévy process, then $X$ is a Feller process.

For $t \geq 0$, $q_t$ denotes the law of $X(t)$, $T_t, t \geq 0$ is the associated Feller semigroup. Then

$$(T_t f)(x) = \int_{\mathbb{R}^N} f(x + y) q_t(dy),$$

where $q_t$ is the law of $X(t)$, $f \in C_b(\mathbb{R}^N), x \in \mathbb{R}^N, t \geq 0$. In other word,

$$(T_t f)(x) = E(f(X(t) + x)).$$
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Semigroup

- **Theorem** Let $X$ be a Lévy process with the Lévy symbol $\eta$ and characteristics $(b, a, \nu)$. Let $(T_t, t \geq 0)$ be the associated Feller semigroup and $A$ be its infinitesimal generator.

- For each $t \geq 0$, $f \in S(\mathbb{R}^N)$, $x \in \mathbb{R}^N$,

$$T_t f(x) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{i(u, x)} e^{t \eta \hat{f}(u)} du,$$

so $T_t$ is a pseudo-differential operator with symbol $e^{t \eta}$.

- For each $f \in S(\mathbb{R}^N)$, $x \in \mathbb{R}^N$,

$$(Af)(x) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{i(u, x)} \eta \hat{f}(u) du,$$

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$$+ \int_{\mathbb{R}^N \setminus \{0\}} [f(x + y) - f(x)y^i \partial_i f(x) \chi_{B_1(0)}(y)] \nu(dy)$$

Examples.

Let $X$ be a standard Brownian motion in $\mathbb{R}^N.$ Then $X$ has characteristics $(0, I, 0),$ so

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Let $X$ be a rotationally invariant stable process of index $\alpha$, $0 < \alpha < 2$. Its symbol $\eta(u) = -|u|^\alpha$ for all $u \in \mathbb{R}^N$. Then

$$A = \eta(D) = -(-\Delta)^\alpha. \quad (u_j \rightarrow -i \partial_j).$$

Fix $m, c > 0$,

$$E_{m,c} = \sqrt{m^2 c^4 + c^2 |u|^2 - mc^2}$$

is a Lévy symbol. Hence,

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Fractional Laplacian and its Extension
The fractional Laplacian of a function \( u : \mathbb{R}^N \to \mathbb{R} \) is expressed by the formula

\[
(\Delta)^\alpha u(x) = C_{N,\alpha} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} \, dy,
\]

which is a nonlocal operator.

L.A.Caffarelli and Silvestre Comm.PDE (2007) use Dirichlet to Neumann mapping to realize the fractional Laplacian \((\Delta)^\alpha\) as the boundary operator of a suitable extension.

Given \( \alpha \in (0, 1) \), consider the space \( H^1_0(\mathbb{R}_+^{N+1}, y^{1-2\alpha}) \) of measurable functions \( v \) with the norm

\[
\| v \|^2_{H^1_0(\mathbb{R}_+^{N+1}, y^{1-2\alpha})} = \int_{\mathbb{R}_+^{N+1}} y^{1-2\alpha} |\nabla v|^2 \, dx \, dy.
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The fractional Laplacian of a function $u : \mathbb{R}^N \to \mathbb{R}$ is expressed by the formula

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L.A. Caffarelli and Silvestre Comm. PDE (2007) use Dirichlet to Neumann mapping to realize the fractional Laplacian $(-\Delta)^\alpha$ as the boundary operator of a suitable extension.

Given $\alpha \in (0, 1)$, consider the space $H_0^1(\mathbb{R}_+^{N+1}, y^{1-2\alpha})$ of measurable functions $v$ with the norm

$$\|v\|_{H_0^1(\mathbb{R}_+^{N+1}, y^{1-2\alpha})}^2 = \int_{\mathbb{R}_+^{N+1}} y^{1-2\alpha} |\nabla v|^2 \, dx \, dy.$$
For \( u \in H^{\alpha}(\mathbb{R}^N) \), the extension \( E_\alpha(u) \) of \( u \) related to the operator \((-\Delta)^\alpha\) is defined as the minimizer of the problem

\[
\min \left\{ \int_{\mathbb{R}_+^{N+1}} y^{1-2\alpha} |\nabla v|^2 \, dx \, dy \mid v \in H^1_0(\mathbb{R}_+^{N+1}, y^{1-2\alpha}), \quad tr_{\mathbb{R}^N} v = u \right\}.
\]
The function $v$ satisfies

$$\text{div}(y^{1-2\alpha} \nabla v) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}, \quad v = u \quad \text{on} \quad \partial \mathbb{R}^{N+1}.$$ 

The fundamental solution of the operator is

$$P_{\alpha}(x, y) = C_{N, \alpha} \frac{y^{2\alpha}}{(|x|^2 + y^2)^{\frac{N+2\alpha}{2}}} , \quad y > 0.$$ 

L.A. Caffarelli and Silvestre showed that

$$\lim_{y \to 0^+} y^{1-2\alpha} \frac{\partial v}{\partial \nu} = (-\Delta)^\alpha u.$$
The function $v$ satisfies

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\]

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\[
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\]
This was proved by the Poisson formula or by using Fourier transform.
The relation was found by M. Stein.\((−\Delta)^\alpha\) is a pseudo-differential operator with the symbol \(|\xi|^\alpha\).
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\((-\Delta)^\alpha\) is a pseudo-differential operator with the symbol \(|\xi|^\alpha\).
Therefore, we may transform the problem

\[ (-\Delta)^\alpha u = f(u) \quad \text{in} \quad \mathbb{R}^N \]

into the problem

\[ \text{div}(y^{1-2\alpha} \nabla v) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}, \]

\[ \lim_{y \to 0^+} y^{1-2\alpha} \frac{\partial v}{\partial \nu} = f(v(x, \cdot)) \quad \text{on} \quad \partial \mathbb{R}^{N+1}. \]
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\]
Let $\Omega \subset \mathbb{N}$ be a bounded domain.

- The problem

$$(-\Delta)^{\alpha} u = f(u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega$$

is not well defined.

- The correct problem is proposed as

$$(-\Delta)^{\alpha} u = f(u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega.$$
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\]
We may define by the spectrum of $-\Delta$ a fractional operator $A^\alpha$.

Let $\{\varphi_k\}_{k=1}^\infty$ denote an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $-\Delta$ in $\Omega$ with homogeneous Dirichlet boundary conditions, associated to the eigenvalues $\{\lambda_k\}_{k=1}^\infty$.

The operator $A^\alpha$ is defined for any $u \in C^\infty_c(\Omega)$ by

$$A^\alpha u = \sum_{k=1}^\infty \lambda_k^\alpha u_k \varphi_k,$$

where

$$u = \sum_{k=1}^\infty u_k \varphi_k, \quad u_k = \int_\Omega u \varphi_k \, dx.$$
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Elliptic Problems with Fractional Laplacian in Bounded Domains

- The operators \((-\Delta)^{\alpha}\) and \(A^{\alpha}\) are not the same, since they have different eigenvalues and eigenfunctions.
- The first eigenvalues of \((-\Delta)^{\alpha}\) is strictly less than the one of \(A^{\alpha}\).
- The problem
  \[
  A^{\alpha} u = f(u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega
  \]
  is proper.
- In the following, we denote \(A^{\alpha}\) by \((-\Delta)^{\alpha}\) in a bounded domain. We consider the problem
  \[
  (-\Delta)^{\alpha} u = f(u), \quad \text{in} \quad \Omega, \quad u = 0, \quad \text{on} \quad \partial\Omega. \tag{3}
  \]
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\[
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\]
The operator \((-\Delta)^\alpha\) can be extended by density for \(u\) in the Hilbert space

\[
H = \{ u \in L^2(\Omega) : \|u\|_H^2 = \sum_{k=1}^{\infty} \lambda_k^\alpha |u_k|^2 < +\infty \}.
\]

We have

\[
H = H_\alpha^\alpha(\Omega) \quad \text{if} \quad \alpha \in (0, \frac{1}{2}); \quad H = H_{00}^{\frac{1}{2}}(\Omega) \quad \text{if} \quad \alpha = \frac{1}{2};
\]

\[
H = H_0^\alpha(\Omega) \quad \text{if} \quad \alpha \in (\frac{1}{2}, 1).
\]
Elliptic Problems with Fractional Laplacian in Bounded Domains

The operator $(-\Delta)^\alpha$ can be extended by density for $u$ in the Hilbert space

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We have

$$H = H^\alpha(\Omega) \text{ if } \alpha \in (0, \frac{1}{2}); \quad H = H^\frac{1}{2}_{00}(\Omega) \text{ if } \alpha = \frac{1}{2}; \quad H = H^\alpha_0(\Omega) \text{ if } \alpha \in (\frac{1}{2}, 1).$$
We will consider the extension problem of (4).

Let \( \Omega \subset \mathbb{N} \) be a bounded domain and \( \mathcal{C} = \Omega \times (0, \infty) \) be a cylinder. Define

\[
H_{0,L}^1(\mathcal{C}) = \{ v \in L^2(\mathcal{C}) \| v \|_{H_{0,L}^1(\mathcal{C})} < \infty \},
\]

where

\[
\| v \|_{H_{0,L}^1(\mathcal{C})} = (K_\alpha \int_{\mathcal{C}} y^{1-2\alpha} |\nabla v|^2 \, dx \, dy)^{\frac{1}{2}}.
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$$H^1_{0,L}(\mathcal{C}) = \{ v \in L^2(\mathcal{C}) \| v \|_{H^1_{0,L}(\mathcal{C})} < \infty \},$$

where

$$\| v \|_{H^1_{0,L}(\mathcal{C})} = (K_\alpha \int_{\mathcal{C}} y^{1-2\alpha} |\nabla v|^2 \, dx \, dy)^{\frac{1}{2}}.$$
For a function $u \in H$, we define the extension $w = E_\alpha(u)$ to the cylinder $C$ as the solution to the problem

$$\text{div}(y^{1-2\alpha}\nabla w) = 0 \quad \text{in} \quad C, \quad w = 0 \quad \text{on} \quad \partial_L C,$$

$$w = u \quad \text{on} \quad \Omega \times \{y = 0\}.$$

The extension operator is an isometry between $H$ and $H_{0,L}^1(C)$. That is

$$\|E_\alpha(u)\|_{H_{0,L}^1(C)} = \|u\|_H, \quad \forall u \in H.$$

We may verify that for $\nu = E_\alpha(u)$,

$$\lim_{y \to 0^+} y^{1-2\alpha} \frac{\partial \nu}{\partial y} = (-\Delta)^\alpha u.$$
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The problem

\[
(-\Delta)_{\alpha} u = f(u), \quad \text{in} \quad \Omega, \quad u = 0, \quad \text{on} \quad \partial \Omega
\]  

(4)

can be transformed into the problem

\[
\text{div}(y^{1-2\alpha} \nabla w) = 0 \quad \text{in} \quad \mathcal{C}, \quad w = 0 \quad \text{on} \quad \partial_{L} \mathcal{C},
\]

\[
\lim_{y \to 0^+} y^{1-2\alpha} \frac{\partial w}{\partial \nu} = f(w(x, \cdot)) \quad \text{on} \quad \Omega \times \{y = 0\}.
\]  

(5)
The problem

\[ (-\Delta)^\alpha u = f(u), \quad \text{in} \quad \Omega, \quad u = 0, \quad \text{on} \quad \partial \Omega \]  \hspace{1cm} (4)

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\[ \text{div}(y^{1-2\alpha}\nabla w) = 0 \quad \text{in} \quad \mathcal{C}, \quad w = 0 \quad \text{on} \quad \partial_L \mathcal{C}, \]

\[ \lim_{y \to 0^+} y^{1-2\alpha} \frac{\partial w}{\partial \nu} = f(w(x, \cdot)) \quad \text{on} \quad \Omega \times \{y = 0\}. \]  \hspace{1cm} (5)
Critical problem of fractional Laplacian
Critical problem of fractional Laplacian

We consider the critical problem

\[ (-\Delta)^\alpha u = u^{\frac{N+2\alpha}{N-2\alpha}} + \lambda u, \quad \text{in} \quad \Omega, \quad u = 0, \quad \text{on} \quad \partial\Omega, \quad (6) \]

where \( 2^* = \frac{2N}{N-2\alpha} \) is the critical exponent of the inclusion \( H^\alpha_0(\Omega) \hookrightarrow L^p(\Omega) \).

The extension problem of (6) is

\[ \text{div}(y^{1-2\alpha}\nabla v) = 0 \quad \text{in} \quad \mathcal{C}, \quad v = 0 \quad \text{on} \quad \partial_L\mathcal{C}, \]

\[ \lim_{y \to 0^+} y^{1-2\alpha} \frac{\partial v}{\partial y} = v^{\frac{N+2\alpha}{N-2\alpha}} + \lambda v \quad \text{on} \quad \Omega \times \{ y = 0 \}. \quad (7) \]
We consider the critical problem

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The functional of problem (7)

\[ I(v) = \frac{1}{2} \int_{C_\Omega} y^{1-2\alpha} |\nabla v|^2 \, dx \, dy - \int_{\Omega \times \{0\}} \left\{ \frac{1}{2\alpha^*} |v|^{2*\alpha} - \frac{\lambda}{2} |v|^2 \right\} \, dx \]

does not satisfy (PS) condition.

| \( I \) satisfies (PS)\(_c\) condition for \( c \in (0, \frac{\alpha}{N} S^\frac{N}{2\alpha}) \), where |
The functional of problem (7)

\[ I(v) = \frac{1}{2} \int_{C_{\Omega}} y^{1-2\alpha} |\nabla v|^2 \, dx \, dy - \int_{\Omega \times \{0\}} \left\{ \frac{1}{2^*} |v|^{2^*_\alpha} - \frac{\lambda}{2} |v|^2 \right\} \, dx \]

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I satisfies (PS)_c condition for \( c \in (0, \frac{\alpha}{N} S^{\frac{N}{2\alpha}}) \), where
For $w \in H^1_0(\mathbb{R}^{N+1})$, $S$ is achieved by $w_\varepsilon = E_\alpha(u_\varepsilon)$, where
\[
 u_\varepsilon(x) = \frac{\varepsilon^{N-2\alpha}}{(|x|^2 + \varepsilon^2)^{\frac{N-2\alpha}{2}}}. 
\]
$S$ is the best constant of the inequality

$$
\int_{\mathbb{R}^{N+1}} y^{1-2\alpha} |\nabla w(x, y)|^2 \, dx \, dy \\
\geq S \left( \int_{\mathbb{R}^N} |w(x, 0)|^{2^{*\alpha}} \, dx \right)^\frac{N-2\alpha}{N}
$$

for $w \in H^1_0(\mathbb{R}^{N+1})$.

$S$ is achieved by $w_\varepsilon = E_\alpha(u_\varepsilon)$, where

$$
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$$

Suppose $0 < \lambda < \lambda_1^\alpha$. For $\alpha = \frac{1}{2}$, Tan showed problem (7) has at least a positive solution, general case was considered by de Pablo et al.
Critical problem of fractional Laplacian

- $S$ is the best constant of the inequality

$$
\int_{\mathbb{R}_+^{N+1}} y^{1-2\alpha} |\nabla w(x, y)|^2 \, dx \, dy 
\geq S\left(\int_{\mathbb{R}^N} |w(x, 0)|^{2^*} \, dx \right)^{N-2\alpha} \frac{N}{N-2\alpha}
$$

for $w \in H_0^1(\mathbb{R}_+^{N+1})$.

- $S$ is achieved by $w_\varepsilon = E_\alpha(u_\varepsilon)$, where

$$
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$$

- Suppose $0 < \lambda < \lambda_1^{\alpha}$. For $\alpha = \frac{1}{2}$, Tan showed problem (7) has at least a positive solution, general case was considered by de Pablo et al.
To verify the mountain pass level \( c \in (0, \frac{\alpha}{N} S(\alpha, N)^{\frac{N}{2\alpha}}) \), we need to use the function \( w_\varepsilon = E_\alpha(u_\varepsilon) \).

However, \( w_\varepsilon \) has no explicitly formula.

But we have the following estimates:

\[
w_\varepsilon(x, y) = \varepsilon^{\frac{2\alpha-N}{2}} w_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right), \quad |\nabla w_1(x, y)| \leq \frac{C}{y} w_1(x, y)
\]

for \( \alpha > 0, (x, y) \in \mathbb{R}_{+}^{N+1} \) which enable us to verify the mountain pass level \( c \in (0, \frac{\alpha}{N} S(\alpha, N)^{\frac{N}{2\alpha}}) \).
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for $\alpha > 0, (x, y) \in \mathbb{R}^{N+1}_+$ which enable us to verify the mountain pass level $c \in (0, \frac{\alpha}{N} S(\alpha, N) \frac{N}{2\alpha})$. 
We consider the existence of infinitely many solutions of the problem

\[
\begin{cases}
(−\Delta)^\alpha u = |u|^{2^*-2}u + \lambda u & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega,
\end{cases}
\]

or equivalently, of the problem

\[
div(y^{1-2\alpha}\nabla v) = 0 \quad \text{in } C, \quad v = 0 \quad \text{on } \partial_L C,
\]

\[
\lim_{y \to 0^+} y^{1-2\alpha} \frac{\partial v}{\partial y} = |v|^{\frac{4\alpha}{N-2\alpha}} v + \lambda v \quad \text{on } \Omega \times \{y = 0\}.
\]
We consider the existence of infinitely many solutions of the problem

\[
\begin{aligned}
(-\Delta)^\alpha u &= |u|^{2^*-2}u + \lambda u \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\] (9)

or equivalently, of the problem

\[
div(y^{1-2\alpha} \nabla v) = 0 \quad \text{in } \mathcal{C}, \quad v = 0 \quad \text{on } \partial_L \mathcal{C},
\]

\[
\lim_{y \to 0^+} y^{1-2\alpha} \frac{\partial v}{\partial y} = |v|^\frac{4\alpha}{N-2\alpha} v + \lambda v \quad \text{on } \Omega \times \{y = 0\}.
\] (10)
Theorem (1 YanYangYu)

If $N > 6\alpha$, then (9) has infinitely many solutions.
The subcritical problem

\[
\begin{align*}
\text{div}(y^{1-2\alpha} \nabla v) &= 0, \quad \text{in } C\Omega, \\
\nu &= 0, \quad \text{on } \partial_L C\Omega, \\
y^{1-2\alpha} \frac{\partial v}{\partial y} &= -|v(x, 0)|^{p_n-1}v(x, 0) - \lambda v(x, 0), \quad \text{on } \Omega \times \{0\}.
\end{align*}
\]

where \(p_n = 2^*_\alpha - \varepsilon_n\) with \(\varepsilon_n \to 0\) possesses infinitely many solutions \(\{v_n^j\}_{j=1}^\infty\). For a fixed \(j\), \(v_n = v_n^j\) is a \((PS)_c\) sequence of the functional \(I\).

- It is not clear if \(\{v_n\}\) has a convergent subsequence.
- Corresponding critical values \(c_n^j\) is increasing in \(j\) and \(c_n^j \to \infty\) as \(j \to \infty\).
The subcritical problem

\[
\begin{cases}
\text{div}(y^{1-2\alpha} \nabla v) = 0, & \text{in } C_\Omega, \\
v = 0, & \text{on } \partial_L C_\Omega, \\
y^{1-2\alpha} \frac{\partial v}{\partial y} = -|v(x, 0)|^{p_n-1} v(x, 0) - \lambda v(x, 0), & \text{on } \Omega \times \{0\}.
\end{cases}
\]

(11)

where \(p_n = 2^*_\alpha - \varepsilon_n\) with \(\varepsilon_n \to 0\) possesses infinitely many solutions \(\{v^j_n\}_{j=1}^\infty\). For a fixed \(j\), \(v_n = v^j_n\) is a \((PS)_c\) sequence of the functional \(I\).

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Critical problem of fractional Laplacian

- The subcritical problem

\[
\begin{aligned}
\begin{cases}
\text{div}(y^{1-2\alpha} \nabla v) = 0, & \text{in } \mathcal{C}_\Omega, \\
v = 0, & \text{on } \partial_L \mathcal{C}_\Omega, \\
y^{1-2\alpha} \frac{\partial v}{\partial y} = -|v(x, 0)|^{p_n-1}v(x, 0) - \lambda v(x, 0), & \text{on } \Omega \times \{0\}.
\end{cases}
\end{aligned}
\]  

(11)

where \( p_n = 2^*_{\alpha} - \varepsilon_n \) with \( \varepsilon_n \to 0 \) possesses infinitely many solutions \( \{v_n^j\}_{j=1}^\infty \). For a fixed \( j \), \( v_n = v_n^j \) is a \((PS)_c\) sequence of the functional \( I \).

- It is not clear if \( \{v_n\} \) has a convergent subsequence.

- Corresponding critical values \( c_n^j \) is increasing in \( j \) and \( c_n^j \to \infty \) as \( j \to \infty \).
We have the following global compactness result:

**Proposition 1** Let \( \{v_n\} \subset H^1_{0,L}(C_\Omega) \) be a solution of (11) satisfying

\[
\|v_n\|_{H^1_{0,L}(C_\Omega)} \leq C.
\]

Then, there exist a solution \( v_0 \in H^1_{0,L}(C_\Omega) \) of (10), a finite sequence \( \{W^j\}_{j=1}^k \subset H^1_{0,L}(\mathbb{R}^N) \) solutions of

\[
\begin{aligned}
\text{div}(y^{1-2\alpha}\nabla v) &= 0, & \text{in } \mathbb{R}^{N+1}_+, \\
y^{1-2\alpha}\frac{\partial v}{\partial y} &= -\beta_j |v(x,0)|^{2\alpha^*-2}v(x,0), & \text{in } \mathbb{R}^N,
\end{aligned}
\] (12)
where $\beta_j \in (0, 1]$ is some constant, and sequences $\{x_j^j\}_{j=1}^k$, $\{\sigma^j_n\}_{j=1}^k$ satisfying $\sigma^j_n > 0$, $x_n^j \in \Omega$ and as $n \to +\infty$,

$$\sigma^j_n \text{dist}(x_n^j, \partial \Omega) \to \infty, \quad \frac{\sigma^j_n}{\sigma^i_n} + \frac{\sigma^i_n}{\sigma^j_n} + \sigma^i_n \sigma^j_n |x_n^i - x_n^j|^2 \to +\infty, \quad i \neq j,$$

(13)

$$\|v_n - v_0 - \sum_{j=1}^k \rho_{x_n^j, \sigma^j_n}(W^j)\|_{H^1_{0,L}(\mathbb{R}^N)} \to 0.$$  \hspace{1cm} (14)

where

$$\rho_{x, \sigma}(W) = \sigma \frac{N-2\alpha}{2} W(\sigma(\cdot - (x, 0))).$$
Theorem (2)

Suppose \( N > 6 \alpha \), then for any \( v_n \), which is a solution of (11) satisfying \( \|v_n\|_{H^1_{0,L}(C_\Omega)} \leq C \) for some constant independent of \( n \), \( v_n \) converges strongly in \( H^1_{0,L}(C_\Omega) \) as \( n \to +\infty \).
Proof of Theorem 2

We will show that there are no bubbles $\rho_{x_n, \sigma_n}(W^j)$ in Proposition 1 appeared.

We argue by contradiction. Suppose the assertion is not true. So there are bubbles $\rho_{x_n, \sigma_n}(W^j)$. Denote by $B_r(z)$ the ball in $\mathbb{R}^{N+1}$, centered at $z \in \mathbb{R}^{N+1}$ with radius $r$,

$$\sigma_n = \min_{1 \leq j \leq k} \sigma^j_n$$

and

$$B_n = B_{t_n \sigma_n^{-\frac{1}{2}}((x_n, 0)) \cap C_\Omega}, \quad \partial_i B_n = \partial B_n \cap C_\Omega.$$
Critical problem of fractional Laplacian

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\]

and

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$$\mathcal{B}_n = B_{t_n^{\sigma_n^{-2}}((x_n, 0)) \cap C_\Omega}, \quad \partial_i \mathcal{B}_n = \partial \mathcal{B}_n \cap C_\Omega.$$
Critical problem of fractional Laplacian

By the Pohozaev identity,

\[
\left( \frac{N}{2} - \frac{N - 2\alpha}{2} \right) \lambda \int_{B_n \cap \{y=0\}} v_n^2 \, dx \\
\leq \int_{(\partial_i B_n) \cap \{y=0\}} \left( \frac{1}{p_n} |v_n|^{p_n} + \frac{1}{2} \lambda v_n^2 \right) \langle x - x_0, \nu_x \rangle \, dS \\
+ \frac{N - 2\alpha}{2} \int_{\partial_i B_n} y^{1-2\alpha} v_n \frac{\partial v_n}{\partial \nu} \, dS \\
- \frac{1}{2} \int_{\partial_i B_n} y^{1-2\alpha} |\nabla v_n|^2 (X - z_0, \nu) \, dS \\
+ \int_{\partial_i B_n} y^{1-2\alpha} (\nabla v_n, X - z_0) \frac{\partial v_n}{\partial \nu} \, dS.
\]
We may deduce that

\[
\text{RHS of (15) } \leq C \sigma_n^{-\frac{1}{2}} \int_{(\partial_i B_n) \cap \{y=0\}} (|v_n|^{p_n} + v_n^2) \, dS \\
+ C \left( \int_{\partial_i B_n} y^{1-2\alpha} |\nabla v_n|^2 \, dS \right)^{\frac{1}{2}} \left( \int_{\partial_i B_n} y^{1-2\alpha} v_n^2 \, dS \right)^{\frac{1}{2}} \\
+ C \sigma_n^{-\frac{1}{2}} \int_{\partial_i B_n} y^{1-2\alpha} |\nabla v_n|^2 \, dS.
\]

(16)
We will show that

$$\text{RHS of (15)} \leq C \sigma_n^{-\frac{N-2\alpha}{2}}.$$

and

$$\int_{\mathcal{B}_n \cap \{y=0\}} v_n^2 \, dx \geq \frac{1}{2} \sigma_n^{-2\alpha},$$

which yields

$$\sigma_n^{-2\alpha} \leq C \sigma_n^{-\frac{N-2\alpha}{2}}$$

a contradiction if $N > 6\alpha$. 
Critical problem of fractional Laplacian


Let $q_1, q_2 \in (2, \infty)$ be such that $q_2 < 2^*_\alpha < q_1$, $\beta > 0$ and $\sigma > 0$. We consider the following inequalities

$$
\begin{align*}
\|u_1\|_{q_1} &\leq \beta, \\
\|u_2\|_{q_2} &\leq \beta \sigma \frac{N}{2^*_\alpha} - \frac{N}{q_2}
\end{align*}
$$

and define the norm

$$
\|u\|_{q_1, q_2, \sigma} = \inf\{\beta > 0 : \text{there exist } u_1, u_2 \text{ such that (17) holds and } |u| \leq u_1 + u_2\}.
$$

Proposition 2 Let $v_n$ be a solution of (11). For any $q_1, q_2 \in \left(\frac{N}{N-2\alpha}, +\infty\right)$, $q_2 < 2^*_\alpha < q_1$, there is a constant $C > 0$, depending only on $q_1$ and $q_2$, such that

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Critical problem of fractional Laplacian

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\|u_1\|_{q_1} \leq \beta, \\
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\[\|v_n\|_{q_1, q_2, \sigma_n} \leq C. \tag{19}\]
Let $w_n > 0$ be the solution of

$$\begin{cases}
\text{div}(y^{1-2\alpha} \nabla w) = 0, & \text{in } C_D, \\
w = 0, & \text{on } \partial_L C_D, \\
y^{1-2\alpha} \frac{\partial w}{\partial \nu} = 2|v_n(x,0)|^{2* - 1} + A, & \text{on } D \times \{0\},
\end{cases} \tag{20}$$

By comparison, $|v_n| \leq w_n$ in $C_{\Omega}$.

It is then sufficient to show

$$\|w_n\|_{q_1,q_2,\sigma_n} \leq C.$$
Critical problem of fractional Laplacian

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By comparison, $|v_n| \leq w_n$ in $C_\Omega$.

It is then sufficient to show

$$\|w_n\|_{q_1,q_2,\sigma_n} \leq C.$$
Lemma 2.2 Let $w_n$ be a solution of (20). There are constants $C > 0$, $q_1, q_2 \in \left(\frac{N}{N-2\alpha}, +\infty\right)$, $q_2 < 2^*_\alpha < q_1$, such that

$$\|w_n\|_{q_1, q_2, \sigma_n} \leq C. \quad (21)$$

Let

$$A^i_n = \{X : X \in \left(\mathcal{B}_{(\tilde{c}+6-i)\sigma_n^{-\frac{1}{2}}(x_n, 0)} \setminus \mathcal{B}_{(\tilde{c}+i-1)\sigma_n^{-\frac{1}{2}}(x_n, 0)}\right) \cap \mathcal{C}_\Omega\},$$

where $i = 1, 2, 3$. 
Lemma 2.2 Let $w_n$ be a solution of (20). There are constants $C > 0$, $q_1, q_2 \in \left(\frac{N}{N-2\alpha}, +\infty\right)$, $q_2 < 2^{*}_\alpha < q_1$, such that

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where $i = 1, 2, 3$. 
To estimate LHS of (16), we note that we may choose $t_n \in [\bar{C} + 2, \bar{C} + 3]$ so that

\[
\begin{align*}
\int_{\partial B_{t_n\sigma_n^{-1/2}}((x_n,0)) \cap C_\Omega} y^{1-2\alpha} \left( \sigma_n^{-1/2} |\nabla v_n|^2 + \sigma_n^{1/2} v_n^2 \right) dS \\
+ \sigma_n^{\alpha - 1/2} \int_{\partial B_{t_n\sigma_n^{-1/2}}((x_n,0)) \cap (\Omega \times \{0\})} (|v_n|^{2^*_\alpha} + v_n^2) dS
\end{align*}
\]

\[
\leq \int_{A_n^3} y^{1-2\alpha} (|\nabla v_n|^2 + \sigma_n v_n^2) \, dx \, dy \\
+ \sigma_n^\alpha \int_{A_n^3 \cap \{y=0\}} (|v_n|^{2^*_\alpha} + v_n^2) \, dx,
\]
Proposition 3 There is a constant $C > 0$, independent of $n$, such that

$$\left( \int_{A_n^2} y^{1-2\alpha} |v_n|^p \, dx \, dy \right)^{\frac{1}{p}} \leq C \sigma_n^{- \frac{N+2-2\alpha}{2p}}$$

(23)

and

$$\int_{A_n^2 \cap \{y=0\}} |v_n|^p \leq C \sigma_n^{- \frac{N}{2}}$$

(24)

for any $p \geq 1$.

Lemma 3.1 Let $w_n$ be a solution of (20). There is a constant, independent of $n$, such that

$$\frac{1}{r^{N+1-2\alpha}} \int_{\partial B_r^+(z) \cap \{y>0\}} y^{1-2\alpha} w_n \, dS \leq C$$

for all $r \geq \bar{C} \sigma_n^{-1/2}$ and $z = (z', 0) \in \Omega$. 
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for all $r \geq \bar{C} \sigma_n^{-1/2}$ and $z = (z', 0) \in \Omega$. 

Let us recall Muckenhoupt class $A_p$ for $p > 1$:

$$A_p = \{ w : \sup_{B} \left( \frac{1}{|B|} \int_{B} |w| \right) \left( \frac{1}{|B|} \int_{B} |w|^{-\frac{1}{p-1}} \right)^{p-1} \leq C, \right.$$

for all ball $B$ in $\mathbb{R}^{N+1} \}.$

It is easy to check that $y^{1-2\alpha} \in A_2$.

Denote $\| u \|_{L^p(E, y^{1-2\alpha})} = \left( \int_{E} y^{1-2\alpha} |u|^p \ dx \right)^{\frac{1}{p}}$. We have the following result, which can be found in E. B. Fabes, C. E. Kenig, R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations 7(1)(1982), 77–116.
Let us recall Muckenhoupt class $A_p$ for $p > 1$:

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Denote $\| u \|_{L^p(E,y^{1-2\alpha})} = \left( \int_{E} y^{1-2\alpha} |u|^p \, dx \right)^{\frac{1}{p}}$. We have the following result, which can be found in E. B. Fabes, C. E. Kenig, R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations 7(1)(1982), 77–116.
Lemma 3.2 Let $\mathcal{D}$ be an open bounded set in $\mathbb{R}^{N+1}$. There exist constants $\delta > 0$ and $C > 0$ depending only on $N$ and $\mathcal{D}$, such that for all $u \in C_0^{\infty}(\mathcal{D})$ and all $k$ satisfying $1 \leq k \leq \frac{N}{N-1} + \delta$,

$$\|u\|_{L^2(\mathcal{D}, y^{1-2\alpha})} \leq C \|\nabla u\|_{L^2(\mathcal{D}, y^{1-2\alpha})}.$$  \hspace{1cm} (25)

Let $D^*$ be an open set in $\mathbb{R}^N$. Consider the following problem:

$$\begin{cases} 
\text{div}(y^{1-2\alpha} \nabla w) = 0, & (x, y) \in C_{D^*}; \\
y^{1-2\alpha} \frac{\partial w}{\partial y} = a(x) w, & x \in D^*, \ y = 0, 
\end{cases} \hspace{1cm} (26)$$

where $a(x) \geq 0$ and $a \in L^\infty_{loc}(\mathbb{R}^N)$. 
**Lemma 3.2** Let $\mathcal{D}$ be an open bounded set in $\mathbb{R}^{N+1}$. There exist constants $\delta > 0$ and $C > 0$ depending only on $N$ and $\mathcal{D}$, such that for all $u \in C_0^\infty(\mathcal{D})$ and all $k$ satisfying $1 \leq k \leq \frac{N}{N-1} + \delta$,

$$
\|u\|_{L^2_\rho(\mathcal{D}, y^{1-2\alpha})} \leq C \|\nabla u\|_{L^2(\mathcal{D}, y^{1-2\alpha})}.
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Let $D^*$ be an open set in $\mathbb{R}^N$. Consider the following problem:

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\begin{cases}
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$$

(26)

where $a(x) \geq 0$ and $a \in L^\infty_{loc}(\mathbb{R}^N)$. 

Critical problem of fractional Laplacian

We have the following estimate:

**Lemma 3.3** Suppose that $w$ is a solution of (26). If there is a small constant $\delta > 0$ such that

$$\int_{B_1(z) \cap \{y=0\}} |a|^\frac{N}{2\alpha} \, dx \leq \delta,$$

for any $B_1(z) \cap \{y = 0\} \subset D^*$, $z = (x, 0)$, then for any $p \geq 1$, there is a constant $C = C(p) > 0$ such that

$$\|w\|_{L^p(B^{+}_{1/2}(z), y^{1-2\alpha})} \leq C \|w\|_{L^1(B^{+}_{1}(z), y^{1-2\alpha})}, \quad (27)$$

and

$$\left(\int_{B^+_r(z) \cap \{y=0\}} w^p \, dx\right)^{\frac{1}{p}} \leq \frac{C}{(R - r)^{\frac{\sigma}{\kappa}}} \|w\|_{L^1(B^+_R(z), y^{1-2\alpha})}, \quad (28)$$

for $p \geq 1$, $0 < \sigma \leq 1$. 
Proposition 3 can be proved by Lemmas 3.1 and 3.3. We also have

Proposition 4

\[ \int_{A_n^3} y^{1-2\alpha} |\nabla v_n|^2 \, dx \, dy \leq C \sigma_n^{-\frac{N-2\alpha}{2}}. \quad (29) \]

Hence, we may deduce (16).

We also have

\[ \int_{B_n \cap \{y=0\}} v_n^2 \, dx \geq \frac{1}{2} \sigma_n^{-2\alpha} \int_{B_1(0) \cap \{y=0\}} W_1^2 + o(\sigma_n^{-2\alpha}). \]
Critical problem of fractional Laplacian

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\]
Fractional Schrödinger Equations in the whole space
The existence of ground state and bound states were considered by Berestycki and P.L. Lions for nonlinear scalar field equations

$$-\Delta u + u = f(u) \quad \text{in } \mathbb{R}^n, \quad u \in H^1(\mathbb{R}^n).$$

(30)

It was shown that there exist a ground state solution and infinitely many bound state solutions of (33) in subcritical superlinear case.

The loss of compactness in $\mathbb{R}^N$ was retained by working in radially symmetric Sobolev spaces.
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The loss of compactness in \( \mathbb{R}^N \) was retained by working in radially symmetric Sobolev spaces.
We consider the existence of ground state and bound state solutions of the equation

$$(−Δ + id)^{1/2} u = f(u) \quad \text{in } \mathbb{R}^n, \quad u \in H^{1/2}(\mathbb{R}^n)$$

(31)

and properties of solutions.
The dynamical behavior of bosons spin-0 particles in relativistic fields can be described by the Schrödinger-Klein-Gordon equation

$$i \frac{\partial \psi}{\partial t} = (-\Delta + id)^{\frac{1}{2}} \psi - \psi + f(x, \psi) \quad \text{in } \mathbb{R}^n. \quad (32)$$

Problem (31) arises in finding the standing wave $e^{it} u(x)$ of the pseudo-relativistic wave equation (32).
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Problem (31) arises in finding the standing wave $e^{it} u(x)$ of the pseudo-relativistic wave equation (32).
Suppose a particle with the mass $m$ moves fast, and the velocity is $v$.

So the energy $E$ and momentum $p$ are given by

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{Ev}{c^2}$$

respectively.

Its Hamiltonian is

$$H = E = \sqrt{m^2 c^4 + p^2 c^2}.$$

Using the momentum operator $p \rightarrow \hbar \nabla$, we obtain

$$\hat{H} = \sqrt{m^2 c^4 - \Delta c^2}.$$

Hence, $\sqrt{-\Delta + id}$ is relativistic, while $\sqrt{-\Delta}$ is non-relativistic.
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  $$H = E = \sqrt{m^2 c^4 + p^2 c^2}.$$  

- Using the momentum operator $p \rightarrow \frac{\hbar}{i} \nabla$, we obtain
  
  $$\hat{H} = \sqrt{m^2 c^4 - \Delta c^2}.$$  

- Hence, $\sqrt{-\Delta + i\hbar}$ is relativistic, while $\sqrt{-\Delta}$ is non-relativistic.
Main results

- **Theorem 1** (TanYangWang) Let $f = |u|^{p-1}u$, where $1 < p < 2^{\#} - 1 = \frac{n+1}{n-1}$, $n \geq 2$. Then, there exists at least one $C^2$ positive ground state solution to problem (31) such that for $\theta \in (0, 1)$,

$$
\lim_{|x| \to \infty} u(x)e^{-\theta|x|} = 0.
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If $p \geq 2^{\#} - 1$, there is no bounded solution of (31).

- **Theorem 2** Let $f(u) = |u|^{p-1}u$, and $1 < p < 2^{\#} - 1 = \frac{n+1}{n-1}$, $n \geq 2$. Then, there exist infinitely many distinct solutions to problem (31).
Main results

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Existence of ground state solutions
- The extension problem related to the operator \((-\Delta + id)^{\frac{1}{2}}\).

For \(u \in H^\frac{1}{2}(\mathbb{R}^n)\), we define the extension of \(u\) related to the operator \((-\Delta + id)^{\frac{1}{2}}\) as the least energy solution \(v \in H^{1,2}(\mathbb{R}_+^{n+1})\) among all finite energy solutions of the problem

\[
\begin{aligned}
-\Delta v(x, y) + v(x, y) &= 0 \quad \text{for} \quad x \in \mathbb{R}^n, \ y > 0, \\
v(x, 0) &= u(x) \quad \text{for} \quad x \in \mathbb{R}^n.
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The extension problem related to the operator \((-\Delta + id)^{1/2}\).

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Lemma 1 Let $u \in H^\frac{1}{2}(\mathbb{R}^n)$. There exists an extension $v$ of $u$. Moreover, $-\partial_y v(x, 0) = (-\Delta + id)^\frac{1}{2} u(x)$.

Hence, problem (31)

$(-\Delta + id)^\frac{1}{2} u = f(u)$ in $\mathbb{R}^n$, $u \in H^\frac{1}{2}(\mathbb{R}^n)$

can be transformed into the problem

$$
\begin{cases}
-\Delta v(x, y) + v(x, y) = 0, & \text{in } \mathbb{R}^{n+1}_+,
\partial v \over \partial y = f(v(x, 0)), & \text{on } \mathbb{R}^n,
\end{cases}
$$

(34)
Lemma 1 Let \( u \in H^{\frac{1}{2}}(\mathbb{R}^n) \). There exists an extension \( v \) of \( u \). Moreover, \(-\partial_y v(x, 0) = (-\Delta + id)^{\frac{1}{2}} u(x)\).

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-\Delta v(x, y) + v(x, y) &= 0, & \text{in} \quad \mathbb{R}^{n+1}, \\
\frac{\partial v}{\partial \nu} &= f(v(x, 0)), & \text{on} \quad \mathbb{R}^n,
\end{aligned}
\] (34)
Proof of Theorem 1 Let us consider the minimizing problem

$$M_p = \inf \left\{ \int_{\mathbb{R}^{n+1}_+} (|\nabla v(x, y)|^2 + |v(x, y)|^2) \, dx dy \mid \int_{\mathbb{R}^n} |v(x, 0)|^{p+1} \, dx = 1 \right\}$$

defined on $H^1(\mathbb{R}^{n+1}_+)$. By the Sobolev trace embedding $H^1(\mathbb{R}^{n+1}_+) \hookrightarrow L^p(\mathbb{R}^n)$ for $n > 2$, we see that the problem $M_p$ is well defined.
Proposition 1 The minimizing problem $M_p$ is achieved by a function $v \in H^1(\mathbb{R}^{n+1}_+)$, which in turn is a solution of problem (33) up to a translation.
The proof of Proposition 1 is based on the following lemma.

**Lemma 2** Let $r > 0$ and $2 \leq q < 2^\# := \frac{2n}{n-1}$. Suppose that \( \{v_m(x, y)\} \) is a bounded sequence in \( H^1(\mathbb{R}^{n+1}_+) \) and that

\[
\sup_{z \in \mathbb{R}^n} \int_{B_r(z)} |v_m(x, 0)|^q \, dx \to 0 \quad (36)
\]

as \( m \to \infty \). Then,

\[
\int_{\mathbb{R}^n} |v_m(x, 0)|^p \, dx \to 0 \quad (37)
\]

as \( m \to \infty \) for \( 2 < p < 2^\# \).
Existence of infinitely many solutions

- By Brezis-Lieb Lemma and Lemma 2, we may show that a minimizing sequence has a convergent subsequence up to translation. Hence, $M_p$ is achieved.

- **Infinitely many solutions.** Let $E = H^1(\mathbb{R}^{n+1}_+)$ and $E_r = \{ v \in H^1(\mathbb{R}^{n+1}_+); v(x, y) = v(|x|, y) \}$.

- Denote by $\mathcal{M} = \{ v \in E_r; \|v\|_{E_r} = 1 \}$ the unit ball in $E_r$. Define the functional

  $$ J(v) = \int_{\mathbb{R}^n} |v(x, 0)|^p \, dx $$

  on $E_r$. 

Existence of ground state solutions
Existence of infinitely many solutions

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We show that $J$ has infinitely many critical points on $\mathcal{M}$.

$J|_{\mathcal{M}}$ satisfies $(PS)_+$ condition in $E_r$, which is proved by

**Lemma 5** Let $2 < q < 2^\# := \frac{2n}{n-1}$ for $n \geq 2$. Then $E_r$ is compactly embedded in $L^q(\mathbb{R}^n)$, where $E_r = \{ v \in H^1(\mathbb{R}_+^{n+1}); v(x, y) = v(|x|, y) \}$. 
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$E_r = \{ v \in H^1(\mathbb{R}^{n+1}_+); v(x, y) = v(|x|, y) \}$. 
Existence of infinitely many solutions

For $k \geq 1$, let $\Gamma_k = \{ A \in \Sigma(\mathcal{M}) : \gamma(A) \geq k \}$, where $\gamma(A)$ is the genus of the set $A$.

Let

$$b_k = \sup_{A \in \Gamma_k} \inf_{w \in A} J(w).$$

Suppose $n \geq 3$, there holds $b_k > 0$ for each $k \geq 1$.

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Exponential Decay

**Exponential Decay.** First, we have

**Proposition 2** Suppose that $f$ satisfies

$$|f(x, s)| \leq C(1 + |s|^p) \quad \text{for all } (x, s) \in \mathbb{R}^n \times \mathbb{R}$$

and $v \in H^1(\mathbb{R}^n)$ is a weak solution of (34). Then $v \in L^q_{loc}(\mathbb{R}_n^{n+1})$ for all $q \in [2, \infty)$. Moreover, $v \in C^{2,\alpha}(\mathbb{R}_n^{n+1})$ and $\lim_{|x| \to \infty} |v(x)| = 0$.

**Local $C^\alpha$ estimate:**

$$\|v(\cdot, 0)\|_{C^\alpha(B_\rho(x))} \leq C(q, \rho)\|v(\cdot, 0)\|_{L^2(B_{2\rho}(x))}$$

which implies $\lim_{|x| \to \infty} |v(x)| = 0$. 

Existence of ground state solutions
Exponential Decay.

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which implies

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Lemma 3 Suppose that there exists $R > 0$ and $w \in H^1(\mathbb{R}^{n+1}_+)$ is a classical solution of

\[
\begin{cases}
-\Delta w + w \geq 0 & \text{in } \mathbb{R}^{n+1}_+, \\
\frac{\partial w}{\partial \nu} \geq \alpha w & \text{in } \mathbb{R}^n \setminus B_R(0), \\
\frac{\partial w}{\partial \nu} \geq 0 & \text{in } B_R(0),
\end{cases}
\]

where $\alpha \in (0, \frac{1}{2})$. Then $w(x, 0) \geq 0$ for $x \in \mathbb{R}^n$. 

Existence of ground state solutions
Lemma 4 There exist $\theta \in (0, 1)$ and $C > 0$ such that

$$u(x) \leq Ce^{-\theta|x|}.$$
Regularity and symmetry

We consider the regularity and symmetry of solutions of the following problem

\[
(-\Delta + id)^{\frac{\alpha}{2}} u = \frac{v^q}{|y|^{2\beta}}, \quad (-\Delta + id)^{\frac{\alpha}{2}} v = \frac{u^p}{|y|^{2\beta}}, \quad \text{in } \mathbb{R}^n, \tag{39}
\]

where \(0 \leq \beta < \alpha < n\), \(1 < p, q < \frac{n-\beta}{\beta}\) and

\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{n - \alpha + \beta}{n}. \tag{40}
\]

Problem

\[
(-\Delta)^{\frac{\alpha}{2}} u = v^q, \quad (-\Delta)^{\frac{\alpha}{2}} v = u^p, \quad \text{in } \mathbb{R}^n. \tag{41}
\]
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Regularity and symmetry

is equivalent to the integral system

\[
  u(x) = \int_{\mathbb{R}^n} \frac{v(y)^q}{|x - y|^{n-\alpha}} \, dy, \quad v(x) = \int_{\mathbb{R}^n} \frac{u(y)^p}{|x - y|^{n-\alpha}} \, dy, \quad \text{in } \mathbb{R}^n.
\]

(42)

The solutions \((u, v)\) of (42) are critical points of the functional associated with the well-known Hardy-Littlewood-Sobolev inequality, which is precisely stated as follows.

**Proposition 3** Let \(0 < \lambda < n\) and let \(1 < p, q < \infty\) such that \(\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2\). Then there holds

\[
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^\lambda} \, dx\, dy \right| \leq C_{q, \lambda, n} \|f\|_p \|g\|_q,
\]

for \(f \in L^p(\mathbb{R}^n)\) and \(g \in L^q(\mathbb{R}^n)\).
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Regularity and symmetry

- Problem (42) is related to the Riesz potentials $I_\alpha(f) = (-\Delta)^{-\frac{\alpha}{2}}$, $0 < \alpha < n$, which is defined by

$$I_\alpha(f)(x) = \frac{1}{C(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy$$

for some $C(\alpha) > 0$. It is known that

$$\|I_\alpha f\|_q \leq C\|f\|_p,$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

- While problem (39) is connected with the Bessel potentials $J_\alpha = (-\Delta + id)^{-\frac{\alpha}{2}}$. The Bessel kernel $G_\alpha$ is given by

$$G_\alpha(x) = \frac{(\sqrt{2\pi})^{-n}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty e^{-s} e^{-\frac{|x|^2}{4s}} s^{\frac{\alpha-n}{2}} \frac{ds}{s}. \quad (43)$$
Problem (42) is related to the Riesz potentials $\mathcal{I}_\alpha(f) = (-\Delta)^{-\frac{\alpha}{2}}$, $0 < \alpha < n$, which is defined by

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Regularity and symmetry

The Hardy-Littlewood-Sobolev inequality for the Bessel potentials with double weights is stated as follows.

**Theorem 3** Let $0 < \alpha < n$, $1 < p, q < \frac{n}{\alpha}$, $\tau, \beta \geq 0$. In addition $n(1 - \frac{1}{p} - \frac{1}{q} + \frac{\alpha}{n}) > \beta + \tau > n(1 - \frac{1}{p} - \frac{1}{q})$. Then, there exists a positive constant $C$ independent of $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ such that the following inequality holds

$$
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) G_\alpha(x - y) h(y) \frac{dxdy}{|x|^{\tau} |y|^{\beta}} \right| \leq C \|f\|_p \|h\|_q. \quad (44)
$$

Furthermore, let

$$Th(x) = \int_{\mathbb{R}^n} \frac{G_\alpha(x - y) h(y)}{|x|^{\tau} |y|^{\beta}} dy,$$

then

$$\|Th\|_{p'} = \sup_{\|f\|_p = 1} |\langle Th, f \rangle| \leq C \|h\|_q. \quad (45)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $1 + \frac{1}{p'} \geq \frac{1}{q} + \frac{n-\alpha+\beta+\tau}{n}$ and $h \in L^q(\mathbb{R}^n)$. 

Existence of ground state solutions
Regularity and symmetry

Theorem 4 If \((u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)\) is a solution pair of (31), then \((u, v) \in L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)\).

Results in Theorem 4 holds also for sign-changing solutions of (39).

In the proof of Theorem 4, we first lift the integrability of a suitable cut-off function of the solution by the regularity lifting method to some \(L^{q_0}\), and then we show that they are actually in \(L^\infty\).
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Let $Z$ be a given vector space, $\| \cdot \|_X$ and $\| \cdot \|_Y$ be two norms on $Z$. Define a new norm $\| \cdot \|_Z$ by

$$\| \cdot \|_Z = \sqrt[p]{\| \cdot \|_X^p + \| \cdot \|_Y^p}.$$  

Suppose that $Z$ is complete with respect to the norm $\| \cdot \|_Z$. Let $X$ and $Y$ be the completion under $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively.

The following regularity lifting theorem was obtained by W. Chen and C. Li.

**Lemma (Regularity Lifting I)** Let $T$ be a contracting map from $X$ into itself and from $Y$ into itself. Assume that $f \in X$ and that there exists a function $g \in Z$ such that $f = Tf + g$, then $f$ also belongs to $Z$.  

Existence of ground state solutions
Let \( Z \) be a given vector space, \( \| \cdot \|_X \) and \( \| \cdot \|_Y \) be two norms on \( Z \). Define a new norm \( \| \cdot \|_Z \) by

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Theorem 5 \( u, v \in C^{0,\gamma}_\text{loc}(\mathbb{R}^n \setminus \{0\}) \), where \( \gamma = 1 - \frac{\beta}{n} \).

Let \( V \) be a Hausdorff topological vector space. Suppose there are two extended norms defined on \( V \),

\[
\| \cdot \|_X, \| \cdot \|_Y : V \to [0, \infty].
\]

Let

\[
X := \{ v \in V : \| v \|_X < \infty \}, \quad Y := \{ v \in V : \| v \|_Y < \infty \}.
\]

The pair of spaces \( (X, Y) \) described as above is called an \( XY - \text{pair} \), if whenever the sequence \( \{ u_n \} \subset X \) with \( u_n \to u \) in \( X \) and \( \| u_n \|_Y \leq C \) will imply \( u \in Y \).
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Lemma (Regularity Lifting II) Suppose that Banach spaces $X, Y$ are an $XY$ – pair, both contained in some larger topological space $V$ satisfying properties described above. Let $\mathcal{X}$ and $\mathcal{Y}$ be closed subsets of $X$ and $Y$ respectively. Suppose that $T : \mathcal{X} \to X$ is a contraction:

$$\| Tf - Tg \|_X \leq \eta \| f - g \|_X, \forall f, g \in \mathcal{X} \text{ and for some } 0 < \eta < 1;$$

and $T : \mathcal{Y} \to Y$ is shrinking:

$$\| Tg \|_Y \leq \theta \| g \|_Y, \forall g \in \mathcal{Y} \text{ and for some } 0 < \theta < 1;$$

Define

$$Sf = Tf + F \text{ for some } F \in \mathcal{X} \cap \mathcal{Y}.$$
Moreover, assume that

\[ S : \mathcal{X} \cap \mathcal{Y} \to \mathcal{X} \cap \mathcal{Y}. \]

Then there exists a unique solution \( u \) of equation

\[ u = Tf + F \text{ in } \mathcal{X}, \]

and more importantly,

\[ u \in Y. \]
Theorem 6 Both $u$ and $v$ are radially symmetric and strictly decreasing about the origin.
Thank you!