Symplectic critical surfaces in Kähler surfaces

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Outline

1. Symplectic surfaces
2. Symplectic critical surfaces
3. Equations and Topological properties
4. The Second variation formula
Let $M$ be a compact Kähler surface, let $\omega$ be the Kähler form.

For a compact oriented real surface $\Sigma$ without boundary which is smoothly immersed in $M$, one defines, following Chern and Wolfson, the Kähler angle $\alpha$ of $\Sigma$ in $M$ as

$$\omega|_{\Sigma} = \cos \alpha \, d\mu_{\Sigma},$$

where $d\mu_{\Sigma}$ is the area element of $\Sigma$. 
As a function on $\Sigma$, $\alpha$ is continuous everywhere and is smooth possibly except at the complex or anti-complex points of $\Sigma$, i.e. where $\alpha = 0$ or $\pi$.

We say that,
$\Sigma$ is a holomorphic curve if $\cos \alpha \equiv 1$,
$\Sigma$ is a Lagrangian surface if $\cos \alpha \equiv 0$,
$\Sigma$ is a symplectic surface if $\cos \alpha > 0$. 
Since

$$\cos \alpha d\mu_{\Sigma} = \omega|_{\Sigma},$$

and

$$d\omega = 0,$$

one gets that

$$l := \int_{\Sigma} \cos \alpha d\mu_{\Sigma}$$

is homotopy invariant.
Recall that the area functional is

\[ A = \int_{\Sigma} d\mu_\Sigma. \]

It is clear that

\[ \cos \alpha \leq 1 \leq \frac{1}{\cos \alpha}, \]

it follows that

\[ \int_{\Sigma} \cos \alpha d\mu_\Sigma \leq A \leq L. \]

We have

\[ l \leq A \leq L. \]
We (Han-Li) consider a new functional:

\[ L_\beta = \int_{\Sigma} \frac{1}{\cos \beta \alpha} d\mu_\Sigma. \]

It is obvious that holomorphic curves minimize the functional if \( \beta > 0. \)
The first variation formula

**Theorem**

Let $M$ be a Kähler surface. The first variational formula of the functional $L_\beta$ is, for any smooth vector field $X$ on $\Sigma$,

\[
\delta_X L_\beta = -(\beta + 1) \int_\Sigma \frac{X \cdot H}{\cos \beta \alpha} d\mu \quad (2.1)
\]

\[
+ \beta (\beta + 1) \int_\Sigma \frac{X \cdot (J (J \nabla \cos \alpha)^\top)) \perp}{\cos \beta + 3 \alpha} d\mu, \quad (2.2)
\]

where $H$ is the mean curvature vector of $\Sigma$ in $M$, and $(\cdot)^\top$ means tangential components of $(\cdot)$, $(\cdot)^\perp$ means the normal components of $(\cdot)$.

The Euler-Lagrange equation of the functional $L_\beta$ is

\[
\cos^3 \alpha H - \beta (J (J \nabla \cos \alpha)^\top)^\perp = 0. \quad (2.3)
\]
Remarks

We call it a $\beta$-symplectic critical surface.

$$\cos^3 \alpha H - \beta (J(J\nabla \cos \alpha)^\top)\perp = 0$$

- $\beta = 0$, we get minimal surface equation;
- If $\beta \to \infty$, we get $\cos \alpha = \text{constant}$.

Proposition

*If a $\beta$-symplectic critical surface is minimal, then $\cos \alpha \equiv \text{Constant}$. 
Let $\{e_1, e_2, v_3, v_4\}$ be a orthonormal frame around $p \in \Sigma$ such that $J$ takes the form

$$J = \begin{pmatrix} 0 & \cos \alpha & \sin \alpha & 0 \\ -\cos \alpha & 0 & 0 & -\sin \alpha \\ -\sin \alpha & 0 & 0 & \cos \alpha \\ 0 & \sin \alpha & -\cos \alpha & 0 \end{pmatrix}.$$ 

Then

$$(J(J\nabla \cos \alpha)^\top)^\perp$$

$$= \cos \alpha \sin^2 \alpha \partial_1 \alpha v_4 + \cos \alpha \sin^2 \alpha \partial_2 \alpha v_3$$

$$= \cos \alpha \sin^2 \alpha (\partial_1 \alpha v_4 + \partial_2 \alpha v_3).$$

Set $\tilde{V} = \partial_2 \alpha v_3 + \partial_1 \alpha v_4.$
Furthermore, we have

\[ \partial_1 \cos \alpha = \omega(\tilde{\nabla}_e e_1, e_2) + \omega(e_1, \tilde{\nabla}_e e_2) \]
\[ = h_1^\alpha \langle Jv_\alpha, e_2 \rangle + h_1^\alpha \langle Je_1, v_\alpha \rangle \]
\[ = (h_1^4 + h_1^3) \sin \alpha. \]

Similarly, we can get that,

\[ \partial_2 \cos \alpha = (h_2^3 + h_1^4) \sin \alpha. \]

Note that

\[ \partial_i \cos \alpha = - \sin \alpha \partial_i \alpha, \quad \text{for } i = 1, 2. \]

Then

\[ \tilde{V} = -(h_2^3 + h_1^4)v_3 - (h_1^4 + h_1^3)v_4. \]
And consequently the Euler-Lagrange equation of the function $L_\beta$ is

$$\cos^2 \alpha H - \beta \sin^2 \alpha V = 0. \quad (2.4)$$

**Proposition**

*For a $\beta$-symplectic critical surface with $\beta \geq 0$, the Euler-Lagrange equation is an elliptic system modulo tangential diffeomorphisms of $\Sigma$.***
Examples in $\mathbb{C}^2$

We consider the $\beta$-symplectic critical surfaces of the following form.

$$F(r, \theta) = (r \cos \theta, r \sin \theta, f(r), 0). \quad (2.5)$$

The equation

$$\cos^3 \alpha H = \beta (J(J \nabla \cos \alpha)^\top)^\perp$$

is equivalent to

$$r(1 + \beta (f')^2)f'' + (1 + (f')^2)f' = 0. \quad (2.6)$$

Set $h = f'$, then (2.6) can be written as

$$r(1 + \beta h^2)h' + (1 + h^2)h = 0, \quad (2.7)$$

which in turn implies that

$$(r(1 + h^2)^{\beta-1} h')' \equiv 0. \quad (2.8)$$
For simplicity, we will consider the special solution of the form

\[ rh(1 + h^2)^{\frac{\beta-1}{2}} \equiv 1, \]  

which implies that for \( r > 0 \)

\[ h(1 + h^2)^{\frac{\beta-1}{2}} = \frac{1}{r} > 0. \]  

- \( \beta = 0 \), we get the catenoid, minimal surface.
- \( \beta = 1 \), we get

\[ F(u, v) = (v \cos u, v \sin u, -\ln v, 0), \]

\[ u \in [0, 2\pi], \quad v > 0. \]

It is in fact a surface \( z = -\frac{1}{2} \log(x^2 + y^2) \) in \( \mathbb{R}^3 \), we consider it as a surface in \( \mathbb{C}^2 \).
- \( \beta \to \infty \), we get the plane.
In fact, for $\beta \geq 1$,

$$\frac{1}{r} = h(1 + h^2)\frac{\beta - 1}{2} \geq h\left(1 + \frac{\beta - 1}{2} h^2\right) \geq \frac{\beta - 1}{2} h^3,$$

we see that for each $r > 0$,

$$0 \leq h(r) \leq \sqrt[3]{\frac{2}{(\beta - 1)r}}. \quad (2.11)$$

This means that $f' = h$ converges to 0 uniformly on each compact subset of $\mathbb{C}^2$. Therefore, we see that $f$ converges to a constant uniformly on each compact subset of $\mathbb{C}^2$. 
We prove a Liouville theorem for $\beta$-symplectic critical surfaces in $\mathbb{C}^2$.

**Theorem**

*If $\Sigma$ is a complete $\beta$-symplectic critical surface in $\mathbb{C}^2$ with area quadratic growth, and $\cos^2 \alpha > \frac{1}{2}$, then it is a holomorphic curve.*
Theorem

If $\Sigma$ is a closed symplectic surface which is smoothly immersed in $M$ with the Kähler angle $\alpha$, then $\alpha$ satisfies the following equation,

$$
\Delta \cos \alpha = \cos \alpha (-|h_{1k}^3 - h_{2k}^4|^2 - |h_{1k}^4 + h_{2k}^3|^2) \\
+ \sin \alpha (H_{,1}^4 + H_{,2}^3) - \frac{\sin^2 \alpha}{\cos \alpha} (K_{1212} + K_{1234}). \quad (3.1)
$$

where $K$ is the curvature operator of $M$ and $H_{,i}^\alpha = \langle \tilde{\nabla}^N_{e_i} H, v_\alpha \rangle$. 

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Theorem

Suppose that $M$ is Kähler surface and $\Sigma$ is a $\beta$-symplectic critical surface in $M$ with Kähler angle $\alpha$, then $\cos \alpha$ satisfies,

$$
\Delta \cos \alpha = \frac{2 \beta \sin^2 \alpha}{\cos \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)} |\nabla \alpha|^2 - 2 \cos \alpha |\nabla \alpha|^2 - \frac{\cos^2 \alpha \sin^2 \alpha}{\cos^2 \alpha + \beta \sin^2 \alpha} Ric(Je_1, e_2).
$$

(3.2)
Corollary

Assume $M$ is Kähler-Einstein surface with scalar curvature $K$, then $\cos \alpha$ satisfies,

$$
\Delta \cos \alpha = \frac{2\beta \sin^2 \alpha}{\cos \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)} |\nabla \alpha|^2 - 2\cos \alpha |\nabla \alpha|^2
$$

$$
- \frac{K \cos^3 \alpha \sin^2 \alpha}{4 \cos^2 \alpha + \beta \sin^2 \alpha}.
$$

(3.3)

Corollary

Any $\beta$-symplectic critical surface in a Kähler-Einstein surface with nonnegative scalar curvature is a holomorphic curve for $\beta \geq 0$. 
By the equations obtained by Micallef-Wolfson, we see that, on a β-symplectic critical surface we have

$$\frac{\partial \sin \alpha}{\partial \zeta} = (\sin \alpha) h,$$

where $h$ is a smooth complex function, $\zeta$ is a local complex coordinate on $\Sigma$, and consequently, we have

**Proposition**

A non holomorphic β-symplectic critical surface in a Kähler surface has at most finite complex points.
Theorem

Suppose that $\Sigma$ is a non holomorphic $\beta$-symplectic critical surface in a Kähler surface $M$. Then

$$\chi(\Sigma) + \chi(\nu) = -P,$$

and

$$c_1(M)([\Sigma]) = -P,$$

where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$, $\chi(\nu)$ is the Euler characteristic of the normal bundle of $\Sigma$ in $M$, $c_1(M)$ is the first Chern class of $M$, $[\Sigma] \in H_2(M, \mathbb{Z})$ is the homology class of $\Sigma$ in $M$, and $P$ is the number of complex tangent points.
Theorem gives a proof of Webster’s formula for $\beta$-symplectic surfaces:

**Corollary**

Suppose that $\Sigma$ is a $\beta$-symplectic critical surface in a Kähler surface $M$. Then

$$\chi(\Sigma) + \chi(\nu) = c_1(M)([\Sigma]).$$
Consider

\[ F_{t,\varepsilon} : \Sigma \times (-\delta, \delta) \times (-a, a) \to M \]

with \( F_{0,0} = F \), where \( F : \Sigma \to M \) is a \( \beta \)-symplectic critical surface. Let

\[
\frac{\partial F_{t,0}}{\partial t} \bigg|_{t=0} = X, \quad \frac{\partial F_{0,\varepsilon}}{\partial \varepsilon} \bigg|_{\varepsilon=0} = Y, \quad \text{and} \quad \frac{\partial^2 F_{t,\varepsilon}}{\partial t \partial \varepsilon} \bigg|_{t=0,\varepsilon=0} = Z.
\]

Denote

\[
\nu_{\beta,t,\varepsilon} = \frac{\det^{(\beta+1)/2}(g_{t,\varepsilon})}{\omega^\beta \left( \frac{\partial F_{t,\varepsilon}}{\partial x^1}, \frac{\partial F_{t,\varepsilon}}{\partial x^2} \right)}
\]

so that

\[
L_\beta(\phi_{t,\varepsilon}) = \int_{\Sigma} \nu_{\beta,t,\varepsilon} dx^1 \wedge dx^2.
\]
It is easy to see that

\[
\frac{\partial}{\partial t} \bigg|_{t=0,\epsilon=0} g_{ij} = \langle \overline{\nabla} e_i X, e_j \rangle + \langle e_i, \overline{\nabla} e_j X \rangle, \tag{4.1}
\]

\[
\frac{\partial}{\partial \epsilon} \bigg|_{t=0,\epsilon=0} g_{ij} = \langle \overline{\nabla} e_i Y, e_j \rangle + \langle e_i, \overline{\nabla} e_j Y \rangle, \tag{4.2}
\]

and

\[
\frac{\partial^2}{\partial t \partial \epsilon} \bigg|_{t=0,\epsilon=0} g_{ij} = \langle \overline{\nabla} e_i Z + \bar{R}(Y, e_i) X, e_j \rangle + \langle e_i, \overline{\nabla} e_j Z + \bar{R}(Y, e_j) X \rangle \\
+ \langle \overline{\nabla} e_i X, \overline{\nabla} e_j Y \rangle + \langle \overline{\nabla} e_i Y, \overline{\nabla} e_j X \rangle. \tag{4.3}
\]

Here, \( \bar{R} \) is the curvature tensor on \( M \).
Assume that $\mathbf{X} = \mathbf{Y}$ is a normal vector field,

$$
\begin{aligned}
&= \frac{\beta + 1}{\cos \beta \alpha} J_0(\mathbf{X}) - \frac{\beta (\beta + 1)}{\cos \beta + 1 \alpha} \nabla_{\nabla \cos \alpha} \mathbf{X} \\
&- \frac{\beta^2 (\beta + 1)^2}{\cos \beta + 6 \alpha} \langle \mathbf{X}, (J(J \nabla \cos \alpha)^\top)^\perp \rangle (J(J \nabla \cos \alpha)^\top)^\perp \\
&- \frac{2 \beta^2 (\beta + 1)}{\cos \beta + 4 \alpha} [\omega(\nabla_{e_1} \mathbf{X}, e_2) + \omega(e_1, \nabla_{e_2} \mathbf{X})] (J(J \nabla \cos \alpha)^\top)^\perp \\
&- \frac{\beta (\beta + 1)}{\cos \beta + 2 \alpha} [\nabla_{e_1} \cos \alpha (J \nabla_{e_2} \mathbf{X})^\perp - \nabla_{e_2} \cos \alpha (J \nabla_{e_1} \mathbf{X})^\perp] \\
&- \beta (\beta + 1) \nabla_{e_1} \frac{\tilde{\omega}(\nabla_{e_1} \mathbf{X}, e_2) + \tilde{\omega}(e_1, \nabla_{e_2} \mathbf{X})}{\cos \beta + 2 \alpha} (Je_2)^\perp \\
&+ \beta (\beta + 1) \nabla_{e_2} \frac{\tilde{\omega}(\nabla_{e_1} \mathbf{X}, e_2) + \tilde{\omega}(e_1, \nabla_{e_2} \mathbf{X})}{\cos \beta + 2 \alpha} (Je_1)^\perp \\
&:= - \int_\Sigma \langle J_\beta \mathbf{X}, \mathbf{X} \rangle d\mu,
\end{aligned}
$$
If we choose $X = x_3 e_3 + x_4 e_4$ and $Y = -J_\nu X = x_4 e_3 - x_3 e_4$, then the second variation formula is

$$II_\beta (X) + II_\beta (Y)$$

$$= -2(\beta + 1) \int_\Sigma \frac{R|X|^2 \sin^2 \alpha}{\cos \beta \alpha} d\mu$$

$$+ (\beta + 1) \int_\Sigma \frac{|\bar{\partial}X|^2 (2\cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^{\beta+2} \alpha} d\mu$$

$$- (\beta + 1) \int_\Sigma \frac{(2\cos^2 \alpha + \beta \sin^2 \alpha)(\cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^{\beta+4} \alpha} |X|^2 |\nabla \alpha|^2 d\mu.$$
As applications of the stability inequality above, we can obtain some rigidity results for stable $\beta$-symplectic critical surfaces.

**Corollary**

Let $M$ be a Kähler surface with positive scalar curvature $R$. If $\Sigma$ is a stable $\beta$-symplectic critical surface in $M$ with $\beta \geq 0$, whose normal bundle admits a nontrivial section $X$ with

\[
\frac{|\bar{\partial}X|^2}{|X|^2} \leq \frac{\cos^2 \alpha + \beta \sin^2 \alpha}{\cos^2 \alpha} |\nabla \alpha|^2,
\]

then $\Sigma$ is a holomorphic curve.
Let $M$ be a Kähler surface with positive scalar curvature $R$. If $\Sigma$ is a stable $\beta$-symplectic critical surface in $M$ with $\beta \geq 0$ and $\chi(\nu) \geq g$, where $\chi(\nu)$ is the Euler characteristic of the normal bundle $\nu$ of $\Sigma$ in $M$ and $g$ is the genus of $\Sigma$, then $\Sigma$ is a holomorphic curve.
we define the set

\[ S : = \{ \beta \in [0, \infty) \mid \exists \text{ strictly stable } \beta - \text{symplectic critical surface } \Sigma \]

\[ \text{with } \int_{\Sigma} |A|^2 d\mu \leq C(s) \} \]

where \( A \) is the second fundamental form of \( \Sigma \) in \( M \), and \( C(s) \) is a positive continuous function.

**Theorem**

*The set \( S \) is open and closed in \([0, \infty)\). In other words, \( S = [0, \infty) \).*

Convergence?

**Conjecture**  *Let \( M \) be a Kähler surface. There is a holomorphic curve in the homotopy class of a symplectic stable minimal surface in \( M \).*

There does exist symplectic stable minimal surfaces which are not holomorphic (Claudi).
Thanks for your attention