Nodal and singular sets for solutions to some elliptic equations

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The nodal set of a function $u$ is defined by

$$N(u) = \{ x : u(x) = 0 \}.$$

The singular set of a harmonic function $u$ is defined by

$$S(u) = \{ x : u(x) = 0, Du(x) = 0 \}.$$
If $\triangle u = 0$ in $B_1 \subseteq R^n$, $u \neq \text{constant}$. Then

$$H^{n-1}(N(u) \bigcap B_r) \leq CN < \infty,$$

$$H^{n-2}(S(u) \bigcap B_r) \leq C(N) < \infty,$$

for $0 < r < 1$, where $N = \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2}$. 
Petrovskii-Oleinik

Let \( f(x), x \in \mathbb{R}^n \), be a polynomial of degree \( N \). Suppose \( \dim_H f^{-1}\{0\} = k \). Then

\[
H^k (f^{-1}\{0\} \cap B_R) \leq c(n) N^{n-k} R^k.
\]
Nodal and singular sets

- To control growth of solutions $\iff$ To control the geometry and topology of their level sets.
- Measure estimates.
- Geometric and topological structures.
Previous works

- In 1979, Almgren first gave the definition of frequency for harmonic functions.
- In 1986, Garofalo and Lin established the monotonicity formula for frequencies and the doubling conditions for solutions of a class of uniformly elliptic linear PDEs.
- Main contributions:
  S.Y.Cheng, H.Donnelly, C.Fefferman, Q.Han, R.Hardt, F-H.Lin, L.Simon, · · · , have studied the frequency functions, growth, nodal sets and singular sets of solutions to elliptic/parabolic equations in $R^n$ and Riemannian manifolds.
Bi-harmonic and H-Harmonic Functions

- **Bi-harmonic functions:** $-\Delta^2 u = 0$.

- **H-harmonic functions:** Let $(z, t) = (x, y, t) \in \mathbb{R}^{2n+1}$, $x, y \in \mathbb{R}^n$,

  \[ -\Delta_H u = \Delta_z u + 4|z|^2 \frac{\partial^2 u}{\partial^2 t} + 4 \frac{\partial}{\partial t} (Pu) = 0, \]

  where $\Delta_z = \sum_{i=1}^n \left( \frac{\partial^2}{\partial^2 x_i} + \frac{\partial^2}{\partial^2 y_i} \right)$ and $Pu = \sum_{i=1}^n \left( y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right) u$.

- **Grushin-harmonic functions:** $Pu = 0$. 
In another form

$$-\Delta_H u = \text{div}(A(z)\nabla u),$$

where

$$A(z) = \begin{pmatrix} I_n & 0_n & (2y)^T \\ 0_n & I_n & (-2x)^T \\ 2y & -2x & 4|z|^2 \end{pmatrix}. $$
H-Harmonic Functions

- $\Delta_H u$ is degenerate.
- The operators $\Delta_H$ are hypoelliptic from Hormander’s hypoellipticity theorem.
- The classical sub-elliptic theory due to G.B. Folland, L.P. Rothschild, E.M. Stein,...
Motivations

- The operators $\Delta_H$ is the sub-Laplacian in the Heisenberg group.
- The Heisenberg group is the simplest model of sub-Riemannian manifolds which are suitable settings of geometric control theory, mathematical physics, CR manifolds and image processing.
The Heisenberg group: $H^n = (R^{2n+1}, \circ)$, where $\circ$ is the group law given by

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(<x', y> - <y', x>))$$

where $x, x', y, y' \in R^n$ and $t, t' \in R$.

$H^n$ is a Lie group with Lie algebra $h_n$ generated by the left-invariant horizontal frame

$$X = \{X_1, \cdots, X_n, X_{n+1}, \cdots, X_{2n}\}$$

given by

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad X_{n+i} = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t} \quad (i = 1, \cdots, n).$$
- **dilation**: \( \delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0. \)

- The Hausdorff dimension of \( H^n \) is \( Q = 2n + 2 \), while the Topology dimension of \( H^n \) is \( N = 2n + 1 \).

- **The gauge norm**:

\[
\rho(z, t) = \left[ \left( \sum_{i=1}^{n} (x_i^2 + y_i^2) \right)^2 + t^2 \right]^{\frac{1}{4}} \equiv \left( |z|^4 + t^2 \right)^{\frac{1}{4}},
\]
The horizontal gradient of a function $u$

$$\nabla_H u = Xu = (X_1 u, \cdots, X_n u, X_{n+1} u, \cdots, X_{2n} u).$$

$$-\Delta_H u = - \sum_{i=1}^{2n} X_i^2 u = 0,$$

which is the Euler equation of (the sub-elliptic) Dirichlet energy functional

$$\frac{1}{2} \int |Xu|^2.$$
Definition

Let $u$ be a bi-harmonic functions in $B_1$ and let $v = \triangle u$. Then we define

$$N(r) = r \frac{\int_{B_r} |\nabla u|^2 + |\nabla v|^2 + uv}{\int_{\partial B_r} u^2 + v^2},$$

$$M(r) = r \frac{\int_{B_r} |\nabla v|^2}{\int_{\partial B_r} v^2}.$$
Definition and Properties of Frequency

- We denote by

\[ D_1(r) = \int_{B_r} |\nabla u|^2, \quad D_2(r) = \int_{B_r} |\nabla v|^2, \quad D_3(r) = \int_{B_r} uv, \]

\[ H_1(r) = \int_{\partial B_r} u^2, \quad H_2(r) = \int_{\partial B_r} v^2, \]

- The functions \( N(r), M(r) \) are the frequencies of \( u \) and \( v \) at the origin with radius \( r \). Similarly, we can define the frequencies of \( u \) and \( v \) at any point \( p \) with radius \( r \), which are denoted by \( N(p, r), M(p, r) \).

- Since \( v \) is harmonic, \( M(r) \) has a lot of interesting properties.
Lemma

Let $v$ be a harmonic function, then

1) if $v$ is a homogeneous harmonic polynomial of degree $k$ then

$$M(r) \equiv k;$$

2) (Monotonicity Formula) $M(r)$ is nondecreasing of $r \in (0, 1);$ 
3) the limit of $M(r)$ as $r \rightarrow 0^+$ exists and is equal to the vanishing order of $v$ at the origin.
Lemma

(4) (Doubling Conditions) for any $R \in (0, 1/2)$ and $\eta \in (1, 2]$

\[
\int_{\partial B_{\eta R}} v^2 \leq \eta^{2M(1)} \int_{\partial B_R} v^2,
\]

\[
\int_{B_{\eta R}} v^2 \leq \eta^{-1} \eta^{2M(1)} \int_{B_R} v^2;
\]

(5) for any $p \in B_R$ with $R < 1$, we have

\[
M(p, \frac{1}{2} (1 - R)) \leq C_1 M(1) + C_2,
\]

where $C_1$ and $C_2$ are positive constants depending only on $n$ and $R$. 

Now let us focus on the frequency of $u$. We can obtain the following properties.

**Lemma**

(1) If the vanishing order of $u$ at the origin is $k \geq 2$, then

$$\lim_{r \to 0} N(r) \geq k - 2.$$

(2)

$$N(r) \geq -C$$

in $B_1$, where $C$ is a positive constant depending only on $n$. 

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Lemma

(3) (Monotonicity Formula) There exists a constant $C_0$ such that if $N(r) \geq C_0$, then

\[
\frac{N'(r)}{N(r)} \geq -C_3 r,
\]

where $C_0$ and $C_3$ are two positive constants depending only on $n$.

(4) For any $p \in B_R$ with $R < 1/2$, we have

\[
N(p, \frac{1}{2}(1 - R)) \leq C_4 N(1) + C_5,
\]

where $C_4$ and $C_5$ are constants depending only on $n$ and $R$. 

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From the monotonicity formula of $N(r)$, we can directly get the following doubling conditions.

**Lemma**

$$\int_{\partial B_{4r}} u^2 + v^2 \leq 2^{C_6 N(1) + C_7} \int_{\partial B_r} u^2 + v^2,$$

and

$$\int_{B_{4r}} u^2 + v^2 \leq 2^{C_8 N(1) + C_9} \int_{B_r} u^2 + v^2,$$

where $r < 1/4$ and $C_6, C_7, C_8$ and $C_9$ are positive constants depending only on $n$. 

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The above doubling conditions are for the integration

\[ \int_{\partial B_r} u^2 + v^2 \quad \text{and} \quad \int_{B_r} u^2 + v^2. \]

But what we need is the doubling conditions for

\[ \int_{\partial B_r} u^2 \quad \text{and} \quad \int_{B_r} u^2. \]

In order to do this, we need the following lemma.
Lemma

For any $r < \frac{1}{4}$, we have

$$r^4 \int_{B_r} v^2 \leq C_{10} 2^{C_{11} M(1)} \int_{B_{2r}} u^2,$$

where $C_{10}$ and $C_{11}$ are positive constants depending only on $n$. 
Doubling Conditions

Sketch of the Proof. For any $\psi \in C_0^\infty(B_1)$,

$$\int_{B_1} \Delta u \Delta \psi = 0.$$ 

Choose $\psi = u\phi^2$, where $\phi$ satisfies

1. $\phi \equiv 1$ in $B_r$ and $\phi \equiv 0$ outside $B_{2r}$;
2. $|\nabla \phi| \leq \frac{C}{r}$ and $|\Delta \phi| \leq \frac{C}{r^2}$.

Then by putting this $\psi$ into the above equation, using the Hölder inequality, integration by parts, Caccippoli inequality of harmonic functions, and doubling conditions of harmonic functions, one can obtain the desired result.
From this lemma and the above doubling conditions, we can obtain the doubling conditions about $\int_{\partial B_r} u^2$ and $\int_{B_r} u^2$.

Lemma

For any $r < \frac{1}{4}$,

$$r^4 \int_{B_{2r}} u^2 \leq 2^{C_{12}(N(1)+M(1))+C_{13}} \int_{B_r} u^2,$$

where $C_{12}$ and $C_{13}$ are positive constants depending only on $n$. 
The measure estimates of bi-harmonic functions are as follows.

**Theorem**

*(L. Tian-Y)* Let $u$ be a bi-harmonic function in $B_1$ and let $v = \triangle u$. Then we have

$$\mathcal{H}^{n-1}\{x \in B_{1/16} | u(x) = 0\} \leq C_{15}(N(1) + M(1)) + C_{16},$$

where $C_{15}$ and $C_{16}$ are positive constants depending only on $n$. 

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Lemma

For any $r \leq \frac{1}{2}$,

$$\sup_{B_r} |u| \leq C_{17}(\|u\|_{L^2(B_{2r})} + r^2 \|v\|_{L^2(B_{2r})}),$$

where $C_{17}$ is a positive constant depending only on $n$. 

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Sketch of the Proof of Theorem

Step 1. From the doubling conditions to show

\[ \int_{B_{1/16}(p)} u^2 \geq 4^{-C_{19}(N(1)+M(1))-C_{20}} \]

for any \( p \in \partial B_{1/4} \), where \( C_{19} \) and \( C_{20} \) are positive constants depending only on \( n \). So there are points \( x_p \in B_{1/16}(p) \) such that

\[ |u(x_p)| \geq 2^{-C_{19}(N(1)+M(1))-C_{20}}. \]
Step 2. We choose \( p_j \) to be the points on \( \partial B_{1/4} \cap j - axis \), \( j = 1, 2, \cdots, n \). Then for any \( w \in S^{n-1} \) and \( j = 1, 2, \cdots, n \), define

\[
 f_j(w; t) = u(x_{p_j} + tw) \quad \text{for} \quad t \in \left(-\frac{5}{8}, \frac{5}{8}\right).
\]

Then \( f_j(w; t) \) is an analytic function of \( t \). Thus we can extend it to be an analytic function \( f_j(w; z) \) of \( z \). Then we obtain

\[
 \mathcal{H}^0 \left\{ |t| < \frac{1}{2} |u(x_{p_j} + tw) = 0 \right\} \leq C_{23}(N(1) + M(1)) + C_{24}.
\]
**Step 3.** Finally from the integral geometric formula, we have

\[ \mathcal{H}^{n-1} \{ x \in B_{1/16} | u(x) = 0 \} \leq C_{25} \sum_{j=1}^{n} \int_{S^{n-1}} N_j(w) \, dw \]

\[ \leq C_{15}(N(1) + M(1)) + C_{16}. \]
The nodal set of H-harmonic polynomial $x^2 - y^2 + t$
The nodal set of H-harmonic polynomial $x^3 + xy^2 + 2ty$. 
Courant’s Nodal Line Theorem: an upper bound for numbers of nodal domains for spherical harmonics.

H. Lewy’s Theorem: minimal number (lower bound) of nodal domains for spherical harmonics, i.e., 2 resp. 3 domains for odd degree $k$ resp. even degree $k$. 
The following homogenous polynomials of degree $k$ are Grushin-harmonic

$$u_{k,l} =: \rho^k \sin^{l/2} \phi \cos^{(l+1)/2} (\cos \phi) e^{i\ell \theta}, \ l = 0, 1, \cdots, k,$$

- There are $\frac{k+l}{2}$ nodal curves of these spherical Grushin-harmonics $u_{k,l}$. 
Nodal domains of spherical Grushin-harmonics

Theorem

(Liu-Tian-Y)

1) For $k \neq 4m$, $m \in \mathbb{N}$, there exists a spherical Grushin harmonic function of degree $k$, such that the nodal lines of this function divide the gauge sphere $S^2$ into two domains.

2) For $k = 4m$, $m \in \mathbb{N}$, there are no spherical Grushin harmonic functions $F$ of degree $k$ such that the nodal curve of $F$ divides the gauge sphere $S^2$ into two parts.
Nodal domains of G-harmonic polynomials

- The nodal domains of G-harmonic polynomials of degree three (two parts)
The nodal domains of G-harmonic polynomials of degree five (two parts)
Nodal domains of G-harmonic polynomials

- The nodal domains of G-harmonic polynomials of degree six (two parts)
Nodal domains of G-harmonic polynomials

The nodal domains of a H-harmonic polynomial of degree five (two parts)
Nodal sets of G-harmonic polynomials

- The nodal domain of G-harmonic polynomial of degree 12 (three parts)
Theorem (Tian-Y)

Suppose that $u$ is a nontrivial $H$–harmonic function in $B_d(0, 1) \subseteq H^n$, and $Pu = 0$. Then there exist positive constants $\tilde{r} < 1$, $r_0$ and $\tilde{r} < r_0$ depending only on $Q$ such that

$$\mathcal{H}^{2n}\{p \in B_d(0, \tilde{r}) : u(p) = 0\} \leq C(N(0, r_0) + 1).$$
Sketch of Proof

Step 1. We claim that

$$N(p, r) \leq CN(0, r_0) + C$$

for $r < cr_0$, where $C$ and $c$ are positive constants depending on $Q$. This claim comes from Lemma of Changing Centers, the monotonicity formula of frequency and the doubling conditions.
Step 2. We first assume that

$$\int_{B_d(0,r_0)} u^2 \psi \, dz \, dt = 1.$$ 

Under this assumption, the doubling condition implies that one can find $p_j$ on the axis, and $p_j \in \partial B_d(0, \frac{r_0}{4})$, $j = 1, 2, \cdots, 2n + 1$, 

$$\int_{B_d(p_j, \frac{r_0}{16})} u^2 \psi \, dz \, dt \geq 4^{-CN(0,r_0) - C},$$

Finally, one can show that there exists $\tilde{p}_j \in B_d(p_j, \frac{r_0}{16})$ such that 

$$|u(\tilde{p}_j)| \geq 2^{-CN(0,r_0) - C}.$$
Step 3. Define \( f_j(\omega; \xi) = u(\tilde{p}_j + \xi \omega) \) for \( \xi \) belongs to suitable interval and \( \omega \) be any unit vector of \( \mathbb{R}^{2n+1} \). Then \( f_j \) are all analytic with respect to \( \xi \). Then we do the complexification of \( f_j \). By using a theorem of H.Donnelly-C.Fefferman, we can have

\[
\mathcal{H}^0 \left\{ |\xi| < \frac{5r_0}{8} : u(\tilde{p}_j + \xi \omega) = 0 \right\} \leq CN(0, r_0) + C.
\]

Step 4. From the integral geometric formula, the desired result can be derived.
Definition of Horizontal Singular Sets

Let $u$ be a smooth function from $H^n$ to $\mathbb{R}$. The horizontal singular set of $u$ is defined as

$$S(u) = \left\{ p \in H^n : u(p) = 0, \sum_{i=1}^{2n} |X_i u|^2(p) = 0 \right\}.$$

In the Heisenberg group, Franchi, Serapioni and Serra Cassano established the implicit function theorem.
We also denote

\[ S_k(u) = \left\{ p \in H^n : X^\alpha u(p) = 0, \forall \alpha \in \bigcup_{m=0}^{k-1} \mathcal{I}_m, \exists \alpha_0 \in \mathcal{I}_k, X^{\alpha_0} u(p) \neq 0 \right\}, \]

where

\[ X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{2n}^{\alpha_{2n}} X_{2n+1}^{\alpha_{2n+1}}, \]

\[ \mathcal{I}_m = \left\{ \alpha = (\alpha_1, \cdots, \alpha_{2n}, \alpha_{2n+1}) : \sum_{i=1}^{2n} \alpha_i + 2\alpha_{2n+1} = m \right\}, \]

and call it the \( k \)-horizontal singular set of \( u \).
Lemma

Let $P$ be a homogeneous polynomial of degree $k$. Then, either

(i) $S_k(P)$ is a linear subspace of $\mathbb{R}^{2n+1}$, and all points on $t$-axis are in $S_k(P)$.

or

(ii) $S_k(P)$ is a linear subspace of $\mathbb{R}^{2n+1}$, and $t$-axis is orthogonal to $S_k(P)$. Moreover, in this case, the dimension of $S_k(P)$ is at most $n$. 
We first prove the following five properties of $S_k(P)$:

1. $0 \in S_k(P)$.
2. $(z, t) \in S_k(P) \Rightarrow \delta_\lambda((z, t)) \in S_k(P)$, $\forall \lambda > 0$.
3. $(z_1, t_1), (z_2, t_2) \in S_k(P) \Rightarrow (z_1, t_1) \circ (z_2, t_2) \in S_k(P)$.
4. $(z, t) \in S_k(P) \Rightarrow (-z, t) \in S_k(P)$.
5. If $(z, t) \in S_k(P)$ for some $t > 0 (t < 0)$, then all points $(0, t)$ satisfying $t > 0 (t < 0)$ are in $S_k(P)$. Moreover, the polynomial $P$ is independent of $t$ in this case.

Then by using this five properties we can get the desired result.
Theorem (Tian-Y)

Let $u$ be a nontrivial $H$-harmonic function in $B_d(0, 1)$. Then the horizontal singular set in $B_d(0, \frac{1}{2})$ is a countable union of $C^1$ sub-manifolds in $\mathbb{R}^{2n+1}$ with dimension at most $2n - 1$. Thus the horizontal singular set of $u$ is at most $(2n - 1)$-countably rectifiable.
Sketch of Proof

Step 1. We first write $S(u)$ as

$$S(u) = \bigcup_{k \geq 2} S_k(u).$$

Because $u$ is a non-trivial $\mathbb{H}$-harmonic function on $\mathbb{H}^n$ and has the strong unique continuity property, that is a finite union.
Step 2. Do the Taylor extension of $u$ at point $z$ for $z \in S_k(u)$. Then

$$u(z \circ p) = P_z(p) + O(d^{k+1}(z^{-1} \circ p)).$$

Let

$$S_k(u) = \bigcup_{j=0}^{2n-1} S_k^j, S_k^j(u) = \overline{S}_k^j(u) \cup \widetilde{S}_k^j(u),$$

where

$$\overline{S}_k^j(u) = \{ z \in S_k(u) : \text{dim} S_k(P_z) = j, P_z \text{ is independent of } t \},$$

$$\widetilde{S}_k^j(u) = \{ z \in S_k(u) : \text{dim} S_k(P_z) = j, P_z \text{ depends on } t \},$$
Step 3. show that $\overline{S}_k^j(u)$ and $\widetilde{S}_k^j(u)$ both are countable union of $j$–dimensional $C^1$–manifolds respectively. That is the result we need.
Corollary

Suppose that $u$ is an $H$-harmonic function in $B_d(0,1) \subseteq H^1$, then the horizontal singular set $S$ of $u$ in $B_d(0,1)$ has the following measure estimate:

$$H^2(S) \leq C < \infty,$$

where $C$ is an absolutely positive constant.
Thank you!