On the rate of convergence in limit theorems for geometric sums of i.i.d. random variables

Tran Loc Hung
University of Finance and Marketing (HCMC, Vietnam)

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(Joint work with Le Truong Giang, UFM, Vietnam)
Main Points

1. Geometric sums.
2. Limit theorems for geometric sums of i.i.d. random variables (The Renyi’s Theorem, 1957).
3. The rate of convergence (o-small rate and O-large rate)
1 For geometric sums of row-wise triangular arrays of independent identically distributed random variables (The Renyi-type theorems)

2 Main tools: The Trotter-operator method, Taylor’s expansion, Geometrically infinitely divisibility, Geometrically strictly stability.
Motivation

1 Geometric Sum:
   - Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) random variables.
   - Let $\nu \sim \text{Geo}(p), p \in (0, 1)$ be a geometric random variable with parameter $p \in (0, 1)$.
   - Denote $S_\nu = X_1 + X_2 + \ldots + X_\nu$ the geometric sum. By convention $S_0 = 0$.

2 Geometric sum has attracted much attention (both in pure and applied maths).


4 Renyi’s Theorem (1957) is one of well-known limit theorems in Probability Theory and related Problems.
References related to Random sum, Geometric sums and Ruin Probability

References related to Geometric sums and Ruin Probability

Classical Renyi Limit Theorem

Renyi’s Theorem, 1957, WLLN

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with mean $0 < m = E(X_1)$

Let $\nu \sim Geo(p), p \in (0, 1)$ be a geometric random variable with success probability $P(\nu = k) = p(1 - p)^{k-1}, k = 1, 2, \ldots$

Denote $S_\nu = X_1 + X_2 + \ldots + X_\nu; S_0 = 0$ the geometric sum.

Let $Z^{(m)}$ be an exponential random variable with positive mean $m$, i.e. $Z^{(m)} \sim \text{Exp}\left(\frac{1}{m}\right)$.

Then

$$pS_\nu \xrightarrow{d} Z^{(m)} \text{ as } p \to 0^+$$
Toda’s Theorem, 2012

Let $X_1, X_2, \ldots$ be a sequence of independent non-identically distributed random variables with mean $0 = E(X_n), 0 < \sigma_n^2 = D(X_n) < \infty; n = 1, 2, \ldots$.

Let $a_j$ be a real sequence such that $a := \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} a_j$ exists.

Let $\lim_{n \to \infty} n^{-\alpha} \sigma_n^2 = 0$ for some $0 < \alpha < 1$ and $\sigma^2 := \frac{1}{n} \sum_{j=1}^{n} \sigma_j^2 > 0$.

for all $\epsilon > 0$, we have an analog of Lindeberg’s condition:

$$\lim_{p \to 0} \sum_{j=1}^{\infty} (1 - p)^{j-1} p E\left[ X_j^2 \left\{ |X_j| \geq \epsilon p^{-\frac{1}{2}} \right\} \right] = 0.$$
Toda’s Theorem, 2012

Theorem

Then

\[ p^{\frac{1}{2}} \sum_{j=1}^{\nu} (X_j + p^{\frac{1}{2}} a_j) \xrightarrow{d} W_{0,a,\sigma} \quad \text{as} \quad p \to 0. \]

where \( W_{0,a,\sigma} \) is an asymmetric Laplace distributed random variable with parameters \((0,a,\sigma)\).

Goal

- The rate of convergence in Renyi-type limit theorems
- The rate of convergence in generalized Renyi-type limit theorems (for negative-binomial random sums).
References related to the Renyi-type limit theorems

References related to the Renyi-type limit theorems

- Samuel Kotz, Tomasz J. Kozubowski and Krzysztof Podgorski, (2001), The Laplace Distribution and Generalizations, Indiana UniversityPurdue University, Indianapolis.
- John Pike and Haining Ren, Stein’s method and the Laplace distribution.
The Trotter’s Operator

**Definition, H. F. Trotter, 1959**

Let $C_B(\mathbb{R})$ be the set of all real-valued bounded and uniformly continuous functions $f$ on $\mathbb{R}$ and $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$. Let $X$ be a random variable. A linear operator $A_X : C_B(\mathbb{R}) \to C_B(\mathbb{R})$, is said to be Trotter operator and it is defined by

$$A_X f(t) := Ef(X + t) = \int_{\mathbb{R}} f(x + t) dF_X(x)$$

where $F_X$ is the distribution function of $X$, $t \in \mathbb{R}$, $f \in C_B(\mathbb{R})$. 

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The Trotter’s Operator

Properties

- The operator $A_X$ is a linear positive ”contraction” operator, i.e.,

  \[ \| A_X f \| \leq \| f \|, \]

  for each $f \in C_B(\mathbb{R})$.

- The operators $A_{X_1}$ and $A_{X_2}$ commute.
The Trotter’s Operator

Properties

- The equation $A_X f(t) = A_Y f(t)$ for $f \in C_B(\mathbb{R}), t \in \mathbb{R}$, provided that $X$ and $Y$ are identically distributed random variables.

- If $X_1, X_2, \ldots, X_n$ are independent random variables, then for $f \in C_B(\mathbb{R})$

$$A_{X_1 + \ldots + X_n}(f) = A_{X_1} \ldots A_{X_n}(f).$$
Properties

Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent random variables (in each group) and they are independent. Then for each $f \in C_B(\mathbb{R})$

$$\|A_{X_1 + \ldots + X_n}(f) - A_{Y_1 + \ldots + Y_n}(f)\| \leq \sum_{i=1}^{n} \|A_{X_i}(f) - A_{Y_i}(f)\|.$$ 

For two independent random variables $X$ and $Y$, for each $f \in C_B(\mathbb{R})$ and $n = 1, 2, \ldots$

$$\|A^n_X(f) - A^n_Y(f)\| \leq n \|A_X(f) - A_Y(t)\|.$$
Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$, are i.i.d. random variables (in each group) and they are independent.

Assume that $\nu \in \text{Geo}(p), p \in (0, 1)$ is geometric distributed random variable, independent of all $X_j$ and $Y_j, j = 1, 2, \ldots$.

Then, for each $f \in C_B(\mathbb{R})$

$$
\| A_{X_1 + \ldots + X_\nu}(f) - A_{Y_1 + \ldots + Y_\nu}(f) \| \leq \frac{1}{p} \| A_{X_1}(f) - A_{Y_1}(f) \|. 
$$
Trotter Operator

Properties

Let

\[
\lim_{n \to \infty} \| A_{X_n}(f) - A_X(f) \| = 0, \forall f \in C^r_B(\mathbb{R}), r \in \mathbb{N}
\]

Then,

\[
X_n \xrightarrow{d} X \text{ as } n \to \infty
\]
References related to the Trotter’s Operator

References related to the Trotter’s Operator

A real-valued random variable $X$ is said to be a geometrically infinitely divisible (g.i.d.) if for any $p \in (0, 1)$, there exists a sequence of real-valued independent identically distributed random variables $X_j(p)$, such that

$$X \overset{d}{=} \sum_{j}^{\nu} X_j(p),$$

where $\nu \sim Geo(p), p \in (0, 1)$, and $\nu$ and all $X_j(q), j = 1, 2, \ldots$ are independent.
Geometrically infinitely divisibility (G.I.D.)

A characteristic function of $X$ is geometrically infinitely divisible if, and only if, it has the form

$$f_X(t) = \frac{1}{1 - \ln \Psi(t)},$$

where $\Psi(t)$ is an infinitely divisibility characteristic function.
Geometrically infinitely divisibility (G.I.D.)

**Theorem, Klebanov, 1984**

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables. Suppose that $\nu$ is a positive integer-valued random variable having geometric distribution with success probability $P(\nu = k) = p(1 - p)^{k-1}, k = 1, 2, \ldots; p \in (0, 1)$. Assume that $\nu$ and $X_1, X_2, \ldots$ are independent. Let us denote by $S_\nu = X_1 + X_2 + \ldots + X_\nu$ the geometric random sum. Suppose that

$$ p \sum_{j=1}^{\nu} X_j(p) \xrightarrow{d} X \quad \text{as} \quad q \to 0. $$

Then, $X$ is geometrically infinitely divisible random variable.
A real-valued random variable $Y$ is said to be a geometrically strictly stability (g.s.s.) if for any $q \in (0, 1)$, there exists a positive constant $c = c(p) > 0$ such that

$$Y \overset{d}{=} c(p) \sum_{j}^{\nu} Y_j(p),$$

where $\nu \sim Geo(p), p \in (0, 1)$, independent of all $Y_j(p), j = 1, 2, \ldots.$
Geometrically strictly stable (G.S.S.)

Theorem, Klebanov, 1984

Let $Y$ be a non-degenerate distributed random variable. The characteristic function of $Y$ is geometrically strictly stable if, and only if, it has the form

$$f_Y(t) = \frac{1}{1 + \lambda |t|^{\alpha} \exp \left( -\frac{i}{2} \theta \alpha \text{sgn}(t) \right)},$$

where $\lambda, \alpha, \theta$ are parameters such that $0 < \alpha \leq 2$, $|\theta| \leq \theta_{\alpha} = \min(1, \frac{2}{\alpha} - 1)$, $\lambda > 0$. 

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Examples

1. the exponential random variable $Z^m \sim \text{Exp}(\frac{1}{m})$ is GID

$$Z^{(m)} \overset{d}{=} p \sum_{j} Z_j$$

where $Z_j \sim \text{Exp}(\frac{1}{m})$

2. the Laplace random variable is GSS

$$W_{0,\sigma} \overset{d}{=} p^{\frac{1}{2}} \sum_{j} W_j,$$
Exponential random variable with mean $m$

1. Density function:

$$p_Z(x) = \frac{1}{m} e^{-\frac{1}{m} x}, \ x \geq 0.$$ 

2. Characteristic function:

$$f_Z(t) = \frac{1}{1 - imt}$$
Symmetric Laplace random variable $W_{0,\sigma}$

1. Density function:

$$p_{W}(x) = \frac{\sigma}{2} \exp^{-\sigma|x|}.$$

2. Characteristic function:

$$f_{W}(t) = \frac{1}{1 + i \frac{1}{\sigma^2} t^2}$$

Note: $W$ is a double exponential random variable.
References related to the G.I.D. and D.S.S.


Lipschitz class

The modulus of continuity of function $f$ is defined for $f \in C_B(\mathbb{R})$, $\delta \geq 0$ by

$$\omega(f; \delta) := \sup_{|h| \leq \delta} \| f(x + h) - f(x) \| .$$  \hspace{1cm} (1)

- The modulus of continuity $\omega(f; \delta)$ is a monotonely decreasing function of $\delta$ with $\omega(f; \delta) \to 0$ for $\delta \to 0^+$, and

  $$\omega(f; \lambda \delta) \leq (1 + \lambda)\omega(f; \delta) \quad \text{for} \quad \lambda > 0.$$

- A function $f \in C_B(\mathbb{R})$ is said to satisfy a Lipschitz condition of order $\alpha$, $0 < \alpha \leq 1$, in symbol $f \in Lip(\alpha)$, if

  $$\omega(f; \delta) = O(\delta^\alpha).$$

  It is easily seen that $f \in Lip(1)$, if $f' \in C_B(\mathbb{R})$. 

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Main Results

Theorem 1

Let \((X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)\) be a row-wise triangular array of non-negative independent identically distributed random variables with
\[ E(|X_{n1}|^{k}) < +\infty, n = 1, 2, \ldots, k = 1, 2, \ldots, r; r = 1, 2, \ldots. \]

Let \(\nu\) be a geometric random variable with parameter \(p, p \in (0, 1)\) and for every \(n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \nu\) are independent.

Moreover, assume that
\[ E|X_{n1}|^{k} = E|Z_{1}^{(m)}|^{k}, k = 1, 2, \ldots, r; r = 1, 2, \ldots. \]
Theorem 1

Then, for \( f \in C^r_B(\mathbb{R}) \)

\[
\| A_{pS_\nu} f - A_{Z(m)} \| = o(p^{r-1}), \quad \text{as} \quad p \to 0.
\]

where \( Z^{(m)} \) is a exponential distributed random variable with positive mean \( E(Z^{(m)}) = m. \)
An analog of Renyi Theorem

**Corollary 1**

- \((X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)\) be a row-wise triangular array of non-negative independent identically distributed random variables with mean 
  \[0 < E(X_{nj}) = m, j = 1, 2, \ldots, n; n = 1, 2, \ldots.\]

- \(\nu\) be a geometric random variable with parameter \(p, p \in (0, 1)\) and for every \(n = 1, 2, \ldots\) \(X_{n1}, X_{n2}, \ldots, \nu\) are independent.

Then,

\[pS_\nu \xrightarrow{d} Z^{(m)}, \quad \text{as} \quad p \to 0\]

where \(S_\nu = X_{n1} + X_{n2} + \ldots + X_{n\nu}, \quad \text{for} \quad n = 1, 2, \ldots\) and \(Z^{(m)}\) is a exponential distributed random variable with positive mean \(E(Z^{(m)}) = m\). Note that \(S_0 = 0\) by convention.
Main Results

**Theorem 3**

- Let \( (X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots) \) be a row-wise triangular array of non-negative valued, independent and identically distributed random variables with finite \( r \)-th absolute moment \( E(|X_{nj}|^r) < +\infty, j = 1, 2, \ldots; r \geq 1 \).
- \( \nu \) be a geometric variable with parameter \( p, p \in (0, 1) \) and \( \nu \) is independent of all \( X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots \).

\[
E(|X_{nj}|^k) = E(|Z_{(m)}^{(j)}|^k); \quad k = 1, 2, \ldots, r - 1; r \geq 1.
\]
Theorem 3

Then, for every $f \in C^{r-1}_B(\mathbb{R})$, 

$$
\| A_{pS} f - A_{Z(m)} f \| \leq \frac{2p^{r-1}}{(r-1)!} \omega(f^{(r-1)};p) \left( m^{r-1}(r-1)! + m^r r! \right)
$$

where $Z_j^{(m)}$ are independent exponential distributed random variables with common mean $m$, i.e. $Z_j^{(m)} \sim Exp\left(\frac{1}{m}\right), j = 1, 2, \ldots$
Main Results

Corollary 2

- \((X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)\) be a row-wise triangular array of non-negative valued, independent and identically distributed random variables with mean \(E(X_{n1}) = m < +\infty\) and finite variance \(0 < D(X_{n1}) = \sigma^2 < +\infty, j = 1, 2, \ldots, n; n = 1, 2, \ldots\).
- \(\nu\) be a geometric variable with parameter \(p, p \in (0, 1)\), \(\nu\) is independent of all \(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots\).

Then, for every \(f \in C^1_B(\mathbb{R})\),

\[ \| A_{p\nu} f - A_{Z(m)} f \| \leq 2 \omega(f'; p) \left( m + \frac{1}{2} \sigma^2 + m^2 \right) \]
Continuous

In particular, suppose that 
\( f' \in \text{Lip}(\alpha, M), \, 0 < \alpha \leq 1, \, 0 < M < +\infty. \) Then

\[
\| A_{pS_{\nu}} f - A_{Z(m)} f \| \leq 2 \left( m + \frac{1}{2} \sigma^2 + m^2 \right) M p^\alpha.
\]
Main Results

Corollary 3

- $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative valued, independent and standard normal distributed random variables.

- $\nu$ be a geometric variable with parameter $p$, $p \in (0, 1)$, and $\nu$ is independent of all $X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots$.

- Denote by $S_{\nu}^2 := X_{n1}^2 + \ldots + X_{n\nu}^2$ by the geometric sum of squared standard normal random variables (another word we call is by chi-squared random variable with geometric degree of freedom)
Corollary 3 (continuous)

Then, for every $f \in C^2_B(\mathbb{R})$,

$$\| A_{pS^2_y} f - A_{Z^{(1)}} f \| \leq \frac{p}{2} \| f'' \| \left( 1 + 24\omega(f''; p) \right),$$

where $Z^{(1)} \sim Exp(1)$. 
Main Results

Theorem 4

Let \((X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)\) be a row-wise triangular array of independent identically distributed random variables with
\[E(|X_{n1}|^k) < +\infty, n = 1, 2, \ldots, k = 1, 2, \ldots, r; r = 1, 2, \ldots.\]

Let \(\nu\) be a geometric random variable with parameter \(p, p \in (0, 1)\) and for every \(n = 1, 2, \ldots\) \(X_{n1}, X_{n2}, \ldots, \nu\) are independent.

Let \(W\) is a Laplace distributed random variable \(W \sim L(0, \sigma)\). Moreover, assume that
\[E|X_{n1}|^k = E|W|^k, k = 1, 2, \ldots, r; r = 1, 2, \ldots\]
Continuous

Theorem 4

Then, for $f \in C_B^r(\mathbb{R})$

$$\| A_{\sqrt{p}S_v} f - A_W f \| = o(p^{r/2-1}), \quad \text{as} \quad p \to 0.$$
Corollary 4

Let \((X_{n,j}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)\) be a row-wise triangular array of independent identically distributed random variables with
\[
\sqrt{p}a = E(X_{n1}); \sigma^2 = D(X_{n1}) < +\infty, n = 1, 2, \ldots.
\]

Let \(\nu\) be a geometric random variable with parameter \(p, p \in (0, 1)\) and for every \(n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \nu\) are independent.

Let \(W\) is a Laplace distributed random variable \(W \sim L(0, \sigma)\).

Then,
\[
\sqrt{p}S_{\nu} \xrightarrow{d} W, \quad \text{as} \quad p \to 0.
\]
Corollary 5

- Let \((X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)\) be a row-wise triangular array of independent identically distributed random variables with \(\sqrt{p}a = E(X_{n1}); \sigma^2 = D(X_{n1}) < +\infty, E|X_{nj}|^3 = \gamma < \infty; n = 1, 2, \ldots.\)

- Let \(\nu\) be a geometric random variable with parameter \(p, p \in (0, 1)\) and for every \(n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \nu\) are independent.

- Let \(W\) is a Laplace distributed random variable \(W \sim L(a, \sigma)\).
Continuous

**Corollary 5**

Then, for every \( f \in C^2_B(\mathbb{R}) \),

\[
\| A \sqrt{p} S_\nu f - A W f \| \leq \frac{\omega(f''; \sqrt{p})}{2} \left[ 2\sigma^2 + \frac{3\sigma^3}{\sqrt{2}} + \gamma \right], \quad \text{as} \quad p \to 0.
\]

If \( f'' \in Lip(\alpha, M) \) with \( 0 < \alpha < 1 \), Then

\[
\| A \sqrt{p} S_\nu f - A W f \| \leq \frac{Mp^{\alpha}}{2} \left[ 2\sigma^2 + \frac{3\sigma^3}{\sqrt{2}} + \gamma \right], \quad \text{as} \quad p \to 0.
\]
Definition

- Let $(X_{n1}, X_{n2}, \ldots)$ be a row-wise triangular array of independent identically distributed random variables.
- Let $\tau$ be a negative-binomial random variable with parameter $l, p, p \in (0, 1), l = 1, 2, \ldots$ such that $P(\tau = k) = C_{k-1}^{l-1}p^lp(1-p)^k - l$. Assume that for every $n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \tau$ are independent.
- Denote by $S_\tau = X_{n1} + X_{n2} + \ldots + X_{n\tau}$ the negative-binomial sum of i.i.d. random variables.
Main Results

Theorem 5

Let \((X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)\) be a row-wise triangular array of independent identically distributed random variables with
\[E(|X_{n1}| = m, n = 1, 2, \ldots, k = 1, 2, \ldots, r; r = 1, 2, \ldots)\]

Let \(\tau\) be a negative-binomial random variable with parameter \(l, p; l \geq 1, p \in (0, 1)\) and for every \(n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \tau\) are independent.

Let \(G\) is a Gamma distributed random variable \(G \sim \Gamma(l, \frac{l}{m})\).
Theorem 5

Then, for every $f \in C^r_B(\mathbb{R})$

$$\| A_p f - A_G f \| = o \left( \left[ \frac{p}{l} \right]^{r-1} \right), \quad \text{as} \quad p \to 0.$$
### Corollary 6 (Generalized Renyi Theorem)

- Let \((X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)\) be a row-wise triangular array of independent identically distributed random variables with 
  \(E(\mid X_{n1} \mid = m, n = 1, 2, \ldots, k = 1, 2, \ldots, r; r = 1, 2, \ldots)\).

- Let \(\tau\) be a negative-binomial random variable with parameter \(l, p; l \geq 1, p \in (0, 1)\) and for every 
  \(n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \tau\) are independent.

- Let \(G\) is a Gamma distributed random variable \(G \sim \Gamma(l, \frac{l}{m})\).

Then,

\[
\frac{p}{l} S_{\tau} \xrightarrow{d} G, \; \text{as} \; p \to 0^+.
\]
Conclusions

- The Trotter method is elementary and elegant (apply to multi-dimensional spaces).
- This method can be applied to a wide class of random variables (not only for continuous class).
- The rates of convergence in limit theorems for geometric sums should be estimated using a probability distance based on Trotter operator

\[
d_A(pS_\nu, Z; f) := \sup_{y \in \mathbb{R}} | E f(pS_\nu + y) - E f(Z + y) |
\]

- Consider the case for geometrical sums of independent non-identically distributed random variables (Toda’s Theorem, 2012)
Thanks for your attention!