EvoLution with Gross Laplacian Noise

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1. Notations, Definitions, Preliminary Results.

\[ \mathcal{H} = L^2(\mathbb{R}_+) \text{ (complex)} \], \( \langle \cdot, \cdot \rangle \) inner product, conjugate linear in the left and linear in the right variables.

The symmetric (Boson) Fock space over \( \mathcal{H} \)

\[ \Gamma(\mathcal{H}) = \left\{ f = \{ f_n \}_{n=0}^{\infty} \left| f_0 \in \mathcal{H}, f_n \in \mathcal{H}^0, (n=1,2 \ldots) \text{ s.t.} \right. \right\} \]

\[ \forall f \in \Gamma(\mathcal{H}) \text{ s.t. } \| f \|^2 = \sum_{n=0}^{\infty} n \| f_n \|^2 < \infty \]

S dense in \( \mathcal{H} \); \( \mathcal{E}(S) \equiv \text{the (dense) linear subspace of } \Gamma(\mathcal{H}) \text{ generated by the exponential (coherent) vectors } \{ \phi_{\frac{1}{2}} | \xi \in \mathcal{S} \} \text{ with } \phi_{\frac{1}{2}} = \left\{ 1, \frac{\xi}{3}, \frac{\xi}{3}, \ldots, (-i)^n \frac{\xi}{3^n}, \ldots \right\} \].

The Annihilation, Creation, Conservation operators are defined as: for \( \xi \in \mathcal{H} \) and \( \text{TeB}(\mathcal{H}) \),

\[ \alpha(\xi) \phi_{\frac{1}{2}} = \langle \xi, \frac{1}{2} \rangle \phi_{\frac{1}{2}}, \]
\[ a^+(a) \varphi_\frac{\lambda}{3} = \frac{d}{d\alpha} \varphi_\frac{\alpha}{3} \bigg|_{\alpha=0}, \quad N(T) \varphi_\frac{\lambda}{3} = \frac{d}{d\alpha} \varphi_\frac{\alpha T^3}{3} \bigg|_{\alpha=0} \]

and observe for \[ \lambda, \rho \in \mathbb{H} \]

\[ \langle \varphi_\lambda, a(\rho) \varphi_\frac{\rho}{3} \rangle = \langle \varphi_\lambda, \varphi_{\rho} \rangle, \quad [a(\rho), a^*(\rho)] = \varphi_{3\rho} \]

\[ \langle \varphi_\lambda, N(T) \varphi_\frac{\rho}{3} \rangle = \langle N(T) \varphi_\lambda, \varphi_{\rho} \rangle = \langle \varphi_\lambda, T \varphi_{3\rho} \rangle \]

\[ \begin{cases} e_{i_1} \cdots e_{i_l} \bigg|_{i_1, \ldots, i_l = 1} \\
\end{cases} \text{a CONB of } \mathbb{H} , \text{ then} \]

\[ N(T) = \sum_{i,j} \langle e_i, T e_j \rangle a^+(e_i) a(e_j), \]

the series converging strongly on \( E(\mathbb{H}) \).

Next, define (generalised) Gross Laplacian

\[ \Delta_{G, T} \equiv \Delta_T \text{ by: } \quad \text{for } T \in B_{2,1}(\mathbb{H}) = \mathbb{H} \otimes \mathbb{H}, \]

\[ \Delta_T \varphi_\frac{\lambda}{3} = \langle T \varphi_\frac{\lambda}{3}, \varphi_\frac{\lambda}{3} \rangle \varphi_\frac{\lambda}{3} = \langle \varphi_{3\lambda}, T \varphi_{3\lambda} \rangle \varphi_\frac{\lambda}{3} \]

equivalently,

\[ \Delta_T = \sum_{i,j} \Delta_{e_i, e_j} a(e_i) a(e_j), \text{ converging strongly on } \mathbb{E}(\mathbb{H}). \]
For such $T$, there is an associated symmetric real kernel $K_T$, i.e., $K_T(x,y) = K_T(y,x) = K_T(x,y)$, and

$$
\Delta_T \phi = \left( \iint K_T(x,y) \phi(x) \phi(y) \, dx \, dy \right) \phi.
$$

For ex. if we take $K_T(x,y) = f(x) f(y)$, a separable kernel then $\Delta_T = a(f)^2$. Also

$$
\langle T \phi, \phi \rangle = \langle T, \phi \phi \rangle = \int k_T(x,y) \overline{\phi(x)} \phi(y) dx dy.
$$

Thus such a $T$ has a 1-1 association with $T \in \mathcal{H}^{(2)}$ and this allows us to define the adjoint of the Gross Laplacian:

$$
\Delta^*_T \phi = \left\{ \phi_T, \phi_T \otimes \phi_T, \phi_T \otimes \phi_T \otimes \phi_T, \ldots \right\},
$$

and we verify that the square of the norm of the R.H.S. is

$$
\| \Delta^*_T \phi \|^2 \leq \sum_{n=0}^{\infty} \frac{(n+2)!}{(n+1)!^2} \left\| \phi \right\|^2 \| \phi \|^2 \leq \infty,
$$

since $\| \phi \| \leq \frac{4n^2}{n^!} \leq \frac{4}{(n-1)!}$ for large $n$. 

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Easy to verify:
\[ \langle \Delta_T^+ \phi_2, \phi_3 \rangle = \langle \phi_2, \Delta T \phi_3 \rangle + \langle \phi_3, \phi_3 \rangle \]
and remark that \( \Delta_T \) in fact, can be defined for every \( T \in \mathcal{B}(\mathcal{H}) \) but \( \Delta_T^+ \) doesn't make sense as an operator in \( \Gamma(\mathcal{H}) \) unless \( T \) is Hilbert-Schmidt.

**Commutation Relations on \( \mathcal{E}(\mathcal{H}) \):**

For \( i = 1, 2, 3 \in \mathcal{H} \) and \( K_i \in \mathcal{B}_2(\mathcal{H}) \), \( L_i \in \mathcal{B}(\mathcal{H}) \), \( i = 1, 2, 3 \in \mathcal{H} \):

(i) \[ \langle a^+(s_i) \phi_2, a^+(s_i) \phi_2 \rangle = \langle \phi_2, s_i \phi_2 \rangle + \langle a(s_i) \phi_2, a(s_i) \phi_2 \rangle \]

(ii) \[ \langle a^+(s_i) \phi_3, N(L_i) \phi_2 \rangle = \langle \phi_3, s_i \phi_2 \rangle + \langle a(L_i^* s_i) \phi_2, a(L_i s_i) \phi_2 \rangle \]

(iii) \[ \langle a^+(s_i) \phi_3, \Delta^+ G \phi_2 \rangle = \langle \Delta G \phi_2, a(s_i) \phi_2 \rangle + \langle a(s_i) \phi_2, a(s_i) \phi_2 \rangle + 2 \langle a(k(s_i) \phi_2, \phi_2 \rangle \]
(v) \[ \langle N(L_1) \Phi_2 \rangle = \frac{1}{2} \left( \langle L_1 \Phi_3 \rangle + \langle L_2 \Phi_3 \rangle \right) \]

(vi) \[ \langle N(L_1) \Phi_2, A^+_G, K_1 \Phi_2 \rangle = \frac{1}{2} \left( \langle A_G, K_1 \Phi_3 \rangle + \langle N(L_1^*) \Phi_2 \rangle \right) + \frac{1}{2} \langle K_1 \Phi_3, \Phi_2 \rangle \]

(vii) \[ \langle A^+_G, K_1 \Phi_3, A^+_G, K_2 \Phi_2 \rangle = \langle A_G, K_2 \Phi_3 \rangle + \langle A_G, K_1 \Phi_2 \rangle + \left\{ 2 \text{Tr}(K_2) + 4 \langle K_1 K_2 \Phi_3, \Phi_2 \rangle \right\} \langle \Phi_3, \Phi_2 \rangle \]

where \( K_1 \Phi_3 = K_3 \Phi_3 \).

These relations follow easily from the definitions and the C.P. R. They can be summarised in the table:

<table>
<thead>
<tr>
<th>( \phi_3 )</th>
<th>( a(\varphi_3) )</th>
<th>( A_G, \phi_3 )</th>
<th>( N(L_2) )</th>
<th>( a(\varphi_2) )</th>
<th>( A^+_G, K_2 )</th>
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<tbody>
<tr>
<td>( \varphi_3 )</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<tr>
<td>( \varphi_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>( \varphi_1 )</td>
<td>( -a(L_1 \varphi_2) )</td>
<td>( -2A_G, J_1 \varphi_1 )</td>
<td>( N(L_1, L_2) )</td>
<td>( a(\varphi_2) )</td>
<td>( 2A^+_G, K_2 )</td>
</tr>
<tr>
<td>( \varphi_1^+ )</td>
<td>( -\langle \varphi_3, \varphi_1 \rangle )</td>
<td>( -2a(T_2 \varphi_1) )</td>
<td>( -a^+(\varphi_2 \varphi_1) )</td>
<td>0</td>
<td>0</td>
</tr>
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<td>( -2a(K_1 \varphi_2) )</td>
<td>( -2\text{Tr}(T_2 K_1) )</td>
<td>( -2A^+_G, L_2 K_1 )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
2. Quantum Stochastic Integrals

The Fock space $\Gamma(\mathfrak{H})$ factorizes as:

$$\mathfrak{H} = \mathfrak{H}_A \otimes \mathfrak{H}_B \otimes \mathfrak{H}_C,$$

where $\mathfrak{H}_A = L^2(\mathcal{A})$; and for a separable initial Hilbert space $\mathcal{H}$ we set

$$\tilde{\mathfrak{H}_p} = \mathcal{H}_p \otimes \Gamma(\mathfrak{H}_p); \quad \tilde{\mathfrak{H}_B} = \mathcal{H}_B \otimes \Gamma(\mathfrak{H}_B); \quad \tilde{\mathfrak{H}_C} = \Gamma(\mathfrak{H}_C).$$

So that $\tilde{\mathfrak{H}_p} \cong \tilde{\mathfrak{H}_B} \otimes \tilde{\mathfrak{H}_C}$ by the map

$$f \otimes \phi_\alpha \leftrightarrow (f \otimes \phi_\alpha) \otimes \phi_\alpha \otimes \phi_\alpha \quad \text{for} \quad f \in \mathcal{H}_p, \quad \alpha \in \mathfrak{H}_B, \quad 0 \leq s < t \leq \alpha.$$

Let $\mathcal{A}$ be dense in $\mathcal{H}$, $M = \{ \xi \in \mathcal{H} \mid \zeta \in \mathcal{B}, \text{ with bounded supp.} \}$

such that $\mathcal{A}, \mathfrak{H}_B \in M \forall \zeta \in \mathcal{M}.$

$\{X_t\}$ is $(\mathcal{A}, M)$-adapted if

(i) $\text{Dom}(X_t) \supset \text{lin} \{ f \otimes \phi_\alpha \mid f \in \mathcal{A}, \quad \alpha \in \mathcal{M} \}$ and

(ii) $X_t(f \otimes \phi_\alpha) = X_t(f \otimes \phi_\alpha) \otimes \phi_\alpha$, for every $0 \leq t \leq \alpha$. 

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Creation, Annihilation and Conservation processes.

For $0 \leq t < \infty$, set $S_t = 1_{[0, \epsilon]} \xi$ and $L_t = L_{1_{[0, \epsilon]}} = 1_{[0, \epsilon]} L \in \mathfrak{B}(\mathcal{H})$, set

$$A_s(t) = \alpha(5_t), \quad A^+_s(t) = \alpha^+(5_t), \quad \Lambda_L(t) = N(L_t);$$

then these are all adapted processes. But they are much more!!

An adapted process $(X_t)_{t \in \mathbb{R}}$ is a Martingale if $t \geq 2, t \in \mathcal{H}, 0 \leq s \leq t < \infty$

$$\langle \frac{\phi}{\infty}, X_t \frac{\phi}{\infty} \rangle = \langle \frac{\phi}{\infty}, X_s \frac{\phi}{\infty} \rangle,$$

and is a process of independent increment

$$\mathbb{L} \left[ X_t \right] = \phi \otimes (X_t - X_s) \phi \otimes \phi \frac{\phi}{\infty},$$

$X_t - X_s$ acts only in $\mathcal{H}(\overline{\mathcal{H}})_{\mathbb{R}}$ non-trivially.

It is easy to verify that the three processes: creation, annihilation, conservation: are all processes of independent increments and hence martingales.
Next, we look at the Gross Laplacians and run into trouble since if we demand naively(!?) like the case of $N(L_t)$ that $K_{[0,1]} = \int_{[0,1]} K \equiv K_{t \wedge T}$, that immediately leads to contradiction with the fact that $K \in \mathcal{B}_2(\mathcal{H})$. Since $\mathcal{H} = L^2(\mathbb{R}^+)$ and since every $K \in \mathcal{B}_2(\mathcal{H})$ there is an integral operator with kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ real symmetric, locally square integrable, i.e. $K(x,z) = K(x,y) = K(y,z)$, $K \in \mathcal{L}_{loc}(\mathbb{R}^2)$ so that we can set $K_t(x,y) = \int_0^t K(x,y) \, dt$ for $0 < t < \infty$ and note that the associated operator (also denoted $K_t$) is in $\mathcal{B}_{2,1}(\mathcal{H})$ has the properties:

a) $\langle K_t \xi, \eta \rangle = \langle K, \xi, \eta \rangle$, \{$K_t$\} is a martingale of Operators in $\mathcal{B}_2(\mathcal{H})$. However, this is \underline{not} a complete martingale.

b) Set $\Delta_t(t) \equiv \Delta K_t^2$, then $\Delta_t(t) = \langle K_t, \xi, \eta \rangle \otimes \xi, \eta \rangle_k^2$

$= (\Delta_t(t) + \xi, \eta)^2 \otimes \xi, \eta$ and similarly
\[ A^+ (t) \otimes \frac{1}{2} = (A^+ (t) \otimes \frac{1}{2} \mathbb{I}) \otimes \frac{1}{2} \mathbb{I} \]

showing that these are adapted processes.

\[ \langle \frac{1}{2} A (t) \otimes \frac{1}{2} \mathbb{I} \rangle = \langle \frac{1}{2} \mathbb{I} \otimes \frac{1}{2} \mathbb{I} \rangle \langle \frac{1}{2} A (t) \otimes \frac{1}{2} \mathbb{I} \rangle \]
\[ = \langle \frac{1}{2} \mathbb{I} \otimes \frac{1}{2} \mathbb{I} \rangle \langle \frac{1}{2} \mathbb{I} \otimes \frac{1}{2} \mathbb{I} \rangle = \langle \frac{1}{2} \mathbb{I} \otimes \frac{1}{2} \mathbb{I} \rangle \]

proving that both \( \{ A (t) \} \) and \( \{ A^+ (t) \} \)

are martingales.

(d) however, it is easy to verify that \( \{ A (t) \} \) and \( \{ A^+ (t) \} \) are NOT processes of independent increment.

The assumption of the property of independent increment is essential to the quantum stochastic integral (see the books of Parthasarathy & Sinha-Goswami).

If we want to include \( \{ A (t) \} \) and \( \{ A^+ (t) \} \) as integrator process, then we'll have to modify the theory.
For simplicity of the rest of the discussion, we assume that \( R(t) = 1 + \gamma t \in \mathbb{R}^+ \). Then write
\[
\Delta_1(t) \equiv \Delta(t) = A(t)^2 \quad \text{and} \quad \Delta^+(t) = A^+(t)^2 \quad \text{where}
\]
\[
A(t) = a\left(e^{\gamma t}\right) \quad \text{and} \quad A^+(t) = a^+\left(e^{\gamma t}\right), \quad \text{for} \quad 0 \leq t < \infty,
\]
so that these Gross Laplacians are squares of the non-commuting quantum Brownian motions, and that they are strongly continuous on \( \mathcal{E}(\mathbb{H}) \).

\[
\Delta(t) - \Delta(s) = 2A(s)J(\mathbb{E}, \mathbb{E}) + J(\mathbb{E}, \mathbb{E})^2, \quad \text{where}
\]
\[
J(\mathbb{E}, \mathbb{E}) = A(t) - A(s) = A(\mathbb{E}, \mathbb{E}) \quad \text{and}
\]
\[
\Delta^+(t) - \Delta^+(s) = 2A^+(s)J^+(\mathbb{E}, \mathbb{E}) + J^+(\mathbb{E}, \mathbb{E})^2, \quad \text{for} \quad 0 \leq s \leq t < \infty.
\]

For various estimates we shall need the following lemma, the proof of which is quite straightforward using the CCR.
Lemma 1: Let \( J(\mathcal{E}, t') \) be as given above. Then

(i) \( J(\mathcal{E}, t')^k + \frac{1}{3} = \left( \int_0^t (\mathcal{E} \, e) \, dt \right)^k \frac{e}{3} \), \( k \in \mathbb{N} \);

(ii) \( \langle J(\mathcal{E}, t')^k + \frac{1}{3}, J(\mathcal{E}, t')^k + \frac{1}{3} \rangle = \left[ (t-s) + (\int_0^t (\mathcal{E} \, e) \, dt)^2 \right] \langle \frac{e}{3}, \frac{e}{3} \rangle \sim (t-s) \),

(iii) \( \langle J(\mathcal{E}, t')^k + \frac{1}{3}, J(\mathcal{E}, t')^{k+2} \rangle = \left[ 2(t-s)(\int_0^t (\mathcal{E} \, e)^2 + \left( \int_0^t (\mathcal{E} \, e)^2 \right)^2 \right] \langle \frac{e}{3}, \frac{e}{3} \rangle \sim (t-s)^{3/2} \),

(iv) \( \langle J(\mathcal{E}, t')^{k+2} + \frac{1}{3}, J(\mathcal{E}, t')^{k+2} + \frac{1}{3} \rangle = \left[ 2(t-s)^2 + 4(t-s)(\int_0^t (\mathcal{E} \, e)^2 + \left( \int_0^t (\mathcal{E} \, e)^2 \right)^2 \right] \langle \frac{e}{3}, \frac{e}{3} \rangle \sim (t-s)^2 \), where \( 0 \leq s \leq t < \infty \),

and \( \frac{e}{3}, \frac{e}{3} \in \mathfrak{H} \).
A Q.S.P. $X$ adapted w.r.t. $(\mathcal{G}, M)$ is simple if for an \( n \searrow \infty \) with $t_0 = 0$, 
\[
X(t) = \sum_{n=0}^{\infty} X(t_n) \mathbf{1}_{[t_n, t_{n+1}]}(t),
\]

and we define Quantum Stochastic Integral (Q.S.I) of $X$ w.r.t. the integrators $M$
$A(t), A^{+}(t), \Lambda(t), \Delta(t), \Delta^{+}(t)$ as:

(i) $\text{Dom}(\Xi(t) = \int_{0}^{t} X \, dM) = \mathcal{D} \otimes \mathcal{E}(M)$,

(ii) for $f \in \mathcal{D}$, $\xi \in M$,
\[
\Xi(t)(f \otimes \xi) = \begin{cases} 
0(t) (f \otimes \xi) \otimes M(t) & \text{if } 0 \leq t \leq t_1; \\
\Xi(t_n) (f \otimes \xi) + X(t_n) [M(t) - M(t_n)] (f \otimes \xi) & \text{if } t_n \leq t \leq t_{n+1} \\
\Xi(t_n) (f \otimes \xi) + X(t_n) \left[ M(t) - M(t_n) \right] \otimes M(t_n) (f \otimes \xi) & \text{if } t = t_n.
\end{cases}
\]

$M = A, A^{+}, \Lambda$;

$= \Xi(t_n) (f \otimes \xi) + 2X(t_n) \Lambda(t_n) (f \otimes \xi) \otimes 
\begin{cases} 
\Xi(t_n) (f \otimes \xi) + X(t_n) (f \otimes \xi) \otimes \mathcal{E}_n & \text{if } t = t_n \\
T(E_n, t) & \text{for } M = A, \Lambda, \Lambda^{+}.
\end{cases}$

\[ -12 - M = A, \text{ and} \]
\[
= \Xi(t_n)(f \otimes \xi) + 2 \times (t_n) A^+(t_n) (f \otimes \xi)
\]
\[
\Xi(t_n) f^+ \xi_n + X(t_n)(f \otimes \xi) \otimes \Xi(t_n) \xi^+ n
\]
if \( M = A^+ \), and

the R.H.S. is defined inductively.

**Lemma 2:** Let \( X \) be as above, and \( M \) be one of \( A^+, A, \Lambda, \Delta, A^+ \) processes. Then for \( \mathbf{f}_G, \mathbf{g}_G, \mathbf{\xi}_n, \mathbf{\xi}_m \):

\[
\langle \mathbf{f}_G \otimes \xi, \int_0^1 X \, dM(\mathbf{f}_G) \rangle = \int_0^1 \langle \mathbf{f}_G \otimes \xi, X(\rho)(\mathbf{g}_G) \rangle \, \mu(\rho) \, d\mu(\rho),
\]

where

<table>
<thead>
<tr>
<th>( M )</th>
<th>( A )</th>
<th>( A^+ )</th>
<th>( \Lambda )</th>
<th>( \Delta )</th>
<th>( A^+ )</th>
</tr>
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<tbody>
<tr>
<td>( M \xi_n )</td>
<td>( \int \mathbf{f}_G \otimes \xi_n , dX )</td>
<td>( \int \mathbf{f}_G \otimes \xi_n , dX^+ )</td>
<td>( \int \mathbf{f}_G \otimes \xi_n , d\Delta )</td>
<td>( \int \mathbf{f}_G \otimes \xi_n , d\Lambda )</td>
<td>( \int \mathbf{f}_G \otimes \xi_n , dA^+ )</td>
</tr>
</tbody>
</table>
Set \( \Xi(t) (f \otimes \phi_{\frac{2}{3}}) = \begin{cases} \frac{t}{t_n} (f \otimes \phi_{\frac{2}{3}}) & 0 < t \leq t_n, \\
\text{same as before for } M = A, A^+, A \\
\text{for } t_{n-1} < t \leq t_n & \text{for } M = A^* & A^+(t_n), \\
2X(t_n) A(t_n) (f \otimes \phi_{\frac{2}{3}}) & \text{for } A^+(t_n). \\
\end{cases} \)

Then

Lemma 3: \( \left\| \Xi_\mathcal{P}(t) - \Xi^\prime_\mathcal{P}(t) \right\| f \otimes \phi_{\frac{2}{3}} \rightarrow 0 \)

as \( |\mathcal{P}| = \max_j (t_j - t_{j-1}) \rightarrow 0. \)

Proof: From the definition it is clear that

\[
\Xi_\mathcal{P}(t) - \Xi_\mathcal{P}^\prime(t) = \sum_{j=1}^{n^+} \left\{ \frac{t}{t_n} \left[ \# \left( t_{j-1}, t_j \right) \right] + \frac{t}{t_n} \left[ \# \left( t_n, t \right) \right] \right\} 
\]

\[
\Xi_\mathcal{P}(t) - \Xi_\mathcal{P}^\prime(t) \left\| f \otimes \phi_{\frac{2}{3}} \right\|^2 \leq \sum_{j=1}^{n^+} \left\{ \left[ \# \left( t_{j-1}, t_j \right) \right] \left( f \otimes \phi_{\frac{2}{3}} \right) \right\}^2 + \left\{ \# \left( t_n, t \right) \right\} \left( f \otimes \phi_{\frac{2}{3}} \right) \right\}^2 
\]

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where we have used the adaptedness of \{X(t)\} and its 'simplicity' to get:

\[
\begin{align*}
\sum \| X(t_{j-1}) \# f [t_{j-1}, t_j] \|^{2} (f \otimes \phi_{3}) \|^{2} \\
\leq \sup_{0 \leq s \leq t} \| X(s) \# f \otimes \phi_{3} \|^{2} \sum \| \# f [t_{j-1}, t_j] \|^{2} \phi_{3}^{2} + \| f \|^{2} \sum (t_j - t_{j-1})^{2}.
\end{align*}
\]

Since by Lemma 1(iv) \| \# f [t_{j-1}, t_j] \|^{2} \phi_{3}^{2} \leq \| f \|^{2} \sum (t_j - t_{j-1})^{2},

Thus we have

\text{Theorem 4: Let } X \text{ be a } (\mathcal{D}, M) \text{-adapted process such that for } f \in \mathcal{D}, \phi \in M,

\text{(i)} \sup_{0 \leq s \leq t} \| X(s) \# f \otimes \phi_{3} \|, \sup_{0 \leq s \leq t} \| X(s) A^+(s) (f \otimes \phi_{3}) \| < \infty \text{ for each } t,

\text{(ii) The maps } s \mapsto X(s) (f \otimes \phi_{3}) \text{ and } s \mapsto X(s) A^+(s) (f \otimes \phi_{3}) \text{ are left continuous. Then }

\int_{0}^{t} X(s) d M(s) (f \otimes \phi_{3}) \text{ exist for } M = A, A^+, A, A^+.
One has furthermore the Quantum Itô formula:

**Theorem 5:** Let \( \{X_j(t)\} \) be two \((\mathcal{F}, M)\)-adapted processes, integrable w.r.t. the martingales \(M_j\) (for \( j = 1, 2 \)) respectively. Then for \( f, g \in \mathcal{D} \) and \( \mathcal{A}, \mathcal{B} \in M_1 \),

\[
\left\langle \int_0^t X_1(s) dM_1(f \otimes \phi_{\mathcal{A}}), \int_0^t X_2(s) dM_2(g \otimes \phi_{\mathcal{B}}) \right\rangle 
\]

\[= \int_0^t \left\langle X_1(s)(f \otimes \phi_{\mathcal{A}}), X_2(s)(g \otimes \phi_{\mathcal{B}}) \right\rangle \, d\mu_{1}(s)\]

\[+ \int_0^t \left\langle \int_0^s X_1(\tau) dM_1(f \otimes \phi_{\mathcal{A}}), X_2(s)(g \otimes \phi_{\mathcal{B}}) \right\rangle \, d\mu_{2}(s)\]

\[+ \int_0^t \left\langle X_1(s) \Theta_1(s)(f \otimes \phi_{\mathcal{A}}), X_2(s) \Theta_2(s)(g \otimes \phi_{\mathcal{B}}) \right\rangle \, d\mu_{12}(s),\]

where

<table>
<thead>
<tr>
<th>(M_1)</th>
<th>(A)</th>
<th>(A^+)</th>
<th>(\mathcal{A})</th>
<th>(\Delta)</th>
<th>(\Delta^+)</th>
</tr>
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<tr>
<td>(M_1)</td>
<td>(s^{8}(0, \mathcal{A}))</td>
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<td>(s^{8}(0, \mathcal{A}))</td>
<td>(s^{8}(0, \mathcal{A}))</td>
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<td>(M_2)</td>
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</tbody>
</table>

and

- \(16\)
\( \mu_{12} \) is given by the table along with \( Q_1 Q_2 \).

<table>
<thead>
<tr>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( A )</th>
<th>( \Delta )</th>
<th>( \Lambda )</th>
<th>( A^+ )</th>
<th>( \Delta^+ )</th>
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<tr>
<td>( A )</td>
<td></td>
<td>0</td>
<td>( \otimes )</td>
<td>2(2) ds, I, I do, I, I</td>
<td>2 ds, I, ( A^+ ) ( t )</td>
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<tr>
<td>( \Delta )</td>
<td></td>
<td>0</td>
<td>( \otimes )</td>
<td>(2(2)) ds, I, I do, I, I do, I, I</td>
<td>4 ds, ( A^+ ) ( t ), ( A^+ ) ( t )</td>
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<td>( \Lambda )</td>
<td>( \otimes )</td>
<td>0</td>
<td>( \otimes )</td>
<td>(2(2)) ds, I, I do, I, I do, I, I do, I, I</td>
<td>(2(2)) ds, I, I do, I, I do, I, I</td>
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<td>( A^+ )</td>
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Another way of reflecting these is to have the table of the product of the differentials as in the next page.
This table summarizes the Quantum Itō calculus rules in this context.

<table>
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<th>$\Delta t$</th>
<th>$d\Delta t$</th>
<th>$2\Delta(t)dt$</th>
<th>$4\Delta(t)A(t)dt$</th>
<th>$4\Delta(t)A(t)$</th>
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<th>$d\Delta$</th>
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Consider the q.s.d.e.

\[ d\Xi = \Xi \left( \sum_{j=1}^{5} X_j \, dM_j + \frac{1}{2} \left( X_6 + X_7 \, A(t) \right) \right. \\
\left. + X_8 \, A^+(t) + X_9 \, A(t) A^+(t) \right) \, dt, \]

with initial value \( \Xi(0) = \Xi_0 \in \mathcal{B}(\mathbb{F}_0) \), and where

\( \{X_j, j=1, \ldots, 9\} \in \mathcal{B}(\mathbb{F}_t) \), and where

\( M_1 = A^+, M_2 = A^-, M_3 = \Lambda, M_4 = A, \)
\( M_5 = \Delta. \)

Theorem 5: The above q.s.d.e. has a unique \( (\mathcal{F}_t, \mathcal{H}) \)-adapted process such that for \( \delta \in \mathbb{H}, T \geq 0, \)

\[ \sup_{0 \leq t \leq T} \sup_{\norm{f}_2 \leq 1} \norm{\Xi(t) (f \otimes \Phi_\delta)} < \infty. \]

The proof, like the earlier ones in the books cited above, follows the standard iteration procedure to arrive at the estimate.
\[ \left\| \mathcal{E}_n(t) - \mathcal{E}_{n-1}(t) \right\|^2 \leq C e^{-n^2/2}, \]

\[ \int_{0 < t_1 < t_2 < \ldots < t_n < t} \left\| \sum_{1 \leq j < k \leq n} X_{i_1} \ldots X_{i_n} \right\|^2, \]

\[ \lambda_\delta(B_\delta) = \alpha(t - \delta) + \beta \int_0^t \| \mathcal{E}(s) \|^2 + \gamma \int_0^t \int_\mathbb{R} \mathcal{E}(s)(x) \mathcal{E}(s)(y) \, dx \, dy, \]

\[ (\alpha, \beta, \gamma \geq 0). \]

**Unitarity of the Solution**

Next, we set the initial value \( \mathcal{E}(0) = \mathcal{E}_0 = I \).

**Theorem 6**: The above q.s.d.e. has a unique unitary evolution (adapted) in \( \mathfrak{U} \otimes \Gamma(\mathbb{H}) \) if and only if

(i) \( X_3 = S - I \) with \( S \) unitary in \( \mathfrak{U} \), and

(ii) \( X_4 = -X_2^* S, X_5 = -X_1^* S \), and

(iii) \( X_6 = -\frac{1}{2} X_2^* X_2 + iH_1, H_1 \in \mathfrak{B}_{s.a.}(\mathfrak{U}) \), and

(iv) \( X_8 = -(X_7^* + 2X_2^* X_1), \) and

(v) \( X_9 = (-4X_1^* X_1 + iH_2), H_2 \in \mathfrak{B}_{s.a.}(\mathfrak{U}). \)
Thus, rewriting the a little differently, the I. S. d. e.

\[ dU(t) = U(t) \left\{ FdA^t + GdA^t + (S - I)dA \right\} 
- G*SdA - F*SdA + \left\{ \left( -\frac{1}{2}G^*G + iH_1 \right) + RA(t) - \left( R + 2G^* \right) \right\} A(t) \left( R + 2G^* \right) dt \right\}, \]

\[ U(0) = 0 \]

has unitary solutions, where \( F, G \in B(\mathbb{F}) \), \( S \in \text{unitary in } H \), \( H_j (j = 1, 2) \in B_{sa}(\mathbb{F}) \).

Strictly speaking, we should have written as a two-variable (time) equation in terms of the evolution \( U_{s, t} (0 \leq s \leq t) \) and the initial value \( U_{s, s} = I \). The above equation is to be interpreted as the right equation in \( t \).

Unfortunately, the usual expectation of \( U_{s, t} (s \leq t) \) does not lead to any "nice" expression except in the two cases:

1. The old familiar one where \( R = F = H_2 = 0 \) and

\[ \langle \phi_0, U_{s, t} \phi_0 \rangle = \exp \left\{ \left( -\frac{1}{2}G^*G + iH_1 \right) (t - s) \right\} \]

2. When \( G = H_1 = R = 0 \) so that

\[ \langle \phi_0, U_{s, t} \phi_0 \rangle = \exp \left\{ \left( -\frac{1}{2}F^*F + iH_2 \right) (t^2 - s^2) \right\}. \]
If we set $F = H_2 = 0$, then we get back the usual Hudson-Parthasarathy equation and the unitary solution, which is an evolution $\Phi$, with its vacuum expectation being a real group, viz.

$$T_{8, t} = \exp \left\{ t \mathcal{L} - \frac{i}{2} G^* G + iH_1 \right\}.$$

This is just the reflection of the property that in such a case the solution $U$ exhibits the "independent increment" property. However, as soon as $F \neq 0$, this "independent increment" property is lost and this is also reflected in the fact that the expectation continues to remain an evolution. In fact, if we set $F = 0$ and $G = H_1 = 0 = H_2$, then

$$T_{4, t} = \exp \left\{ -4 F^* F \left( t^2 - s^2 \right) \right\}.$$
This also implies that the unitary solution $U_{s,t}$ is not a cocycle, though an evolution (random and quantum!).