Itô formula for generalized white noise functionals

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Outline

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6. Complex Brownian functionals
7. Itô formula for Complex Brownian Motion
8. Itô formula for fractional Brownian motion on the classical Wiener space
Without the definition of Itô integral, we are able to derive ”Itô’s formula”, in the proof we show that the Hitsuda-Skorokhod integral arises naturally. In this talk we shall show that the Hitsuda-Skorokhod integral may be defined for any Gaussian and Non-Gaussian Lévy functionals. The main idea was initiated from the following papers:

Basic Notations

- \( S \): the Schwartz space
- \( S' \): the space of tempered distribution
- \((\cdot, \cdot)\): the \( S' - S \) pairing
- \( S_0 = L^2(\mathbb{R}) \)
- \( A : Au = -u'' + 1 + u^2 \), \( A \) is densely defined in \( S_0 \)
- \( \{ e_j : j = 0, 1, 2, \ldots \} \): CONS of \( S_0 \), consisting of eigenfunctions of \( A \) with corresponding eigenvalues \( \{2j + 2 : j = 0, 1, 2, \ldots \} \)
- \( S_p = \{ f \in S' : \| f \|_p < \infty \} \) where

\[
\| f \|_p^2 = \sum_{j=0}^{\infty} (2j + 2)^p(f, e_j)^2.
\]
Basic Notations, cont.

- \( S = \cap_{p \geq 0} S_p; \quad S' = \cup_{p \geq 0} S_{-p} \)
- \( S \subset H \subset S' \) forms a Gel'fand triple.
- \( \mu: \) a standard Gaussian measure defined on \((S', \mathcal{B}(S'))\) with the characteristic functional \( C \) on \( S \) given by
  \[
  C(\eta) = \int_{S'} e^{(x,\eta)} \, \mu(dx) = e^{-\frac{1}{2} \|\eta\|_0^2}
  \]
  where \( \|\eta\|_0 = \left\{ \int_{-\infty}^{+\infty} \eta(t)^2 \, dt \right\}^{1/2} \) (\( \eta \in S \)).
- \((L^2) := L^2(S', \mu)\)
Wiener-Itô decomposition theorem

For \( f \in (L^2) \), \( f \) enjoys the following orthogonal decomposition

\[
    f(x) = \sum_{n=0}^{\infty} \bigoplus \left\{ \frac{1}{n!} \int_{S'} D^n \mu f(0)(x + iy)^n \mu(dy) \right\}
\]

where \( \mu f = \mu \ast f \) and we have

\[
    \|f\|_{L^2(S,\mu)}^2 = \sum_{N=0}^{\infty} \frac{1}{N!} \|D^N \mu f(0)\|_{HS^n(S_0)}^2.
\]

where \( \|T\|_{HS^n(V)} \) denotes the Hilbert-Schmidt norm of the \( n \)-linear operator \( T \) on the Hilbert space \( V \).
Let $f_n$ denote the kernel of $D^n \mu f(0)/n!$ and $B(t)$ denote the Brownian motion, then

$$I_n(f_n) := \int \ldots \int_{\mathbb{R}^n} f_n(t_1, \ldots, t_n) dB(t_1, x) \ldots dB(t_n, x)$$

$$= \frac{1}{n!} \int_{S'} D^n \mu f(0)(x + iy)^n \mu(dy) \text{ a.e.}(\mu)$$

In notation, we write $f \sim (f_n)$. 
The Segal-Bargmann transform of Gaussian WNF

For $\xi \in S$ and for $f \in (L^2) = L^2(S', \mu)$ with $f \sim (f_n)$, define the transform $S$ on $(L^2)$ by

$$S(f)(\xi) = \sum_{n=0}^{\infty} \int \ldots \int_{\mathbb{R}^n} f_n(t_1, \ldots, t_n)\xi(t_1)\ldots\xi(t_n)dt_1 \ldots dt_n$$

or,

$$S(f)(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n \mu f(0)\xi^n.$$ 

Clearly,

$$S(f)(\xi) = \mu * f(\xi) = \int_{S'} f(x + \xi)\mu(dx) = e^{-\frac{1}{2}\|\xi\|^2_0} \int_{S'} f(x)e^{(x,\xi)}\mu(dx).$$
The test functionals

Let \((S)_p\) denote the collection of functions \(f\) such that

\[
\|f\|_{2,p} = \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left\| D^n \mu f(0) \right\|_{HS^n(S_{-p})}^2 \right\}^{1/2} < \infty.
\]

\((S)_{-p}\) is identified as the dual of \((S)_p\). For \(p > q\)

\((S)_p \subset (S)_q \subset (L^2) \subset (S)_{-q} \subset (S)_p\).

Set \((S) = \cap_{p \geq 0}(S)_p\) (with projective limit topology). The we have

\((S) \subset (L^2) \subset (S)^*\)

forms a Gel’fand triple. Members of \((S)\) are called test (Gaussian) white noise functionals and members of its dual space \((S)^*\) are called generalized white noise functionals.
Analyticity of test functionals

Theorem

[L, 1991] For any \( f \in (S) \), there exist an analytic function \( \tilde{f} \) defined on the complexification \( CS' \) such that \( f = \tilde{f} \) a.e. \((\mu)\). Moreover, for each \( p \geq 0 \), there exist a constant \( C_f \), depending only on \( f \), such that

\[
|\tilde{f}(z)| \leq C_f e^{\frac{1}{2}\|z\|^2 - p}.
\]

In what follow we identify \( \varphi \) with \( \tilde{\varphi} \) for any \( \varphi \in (S) \).
An analytic version of \((S)\)

For \(p \in \mathbb{R}^1\), denote by \(A_p\) the class of entire functions \(f\) defined on \(S_{-p}\) which has an entire extension \(\tilde{f}\) to \(\mathcal{C}S_{-p}\) such that

\[
\|f\|_{A_p} := \sup_{z \in \mathcal{C}S_{-p}} \{ |\tilde{f}(z)| e^{-\frac{1}{2} \|z\|_{-p}^2} \} < \infty.
\]

In the sequel we shall identify \(f\) with \(\tilde{f}\) for \(f \in A_p\).
The space $\mathcal{A}_\infty$

Let $\mathcal{A}_\infty = \cap_{p>0} \mathcal{A}_p$. Endow $\mathcal{A}_\infty$ with the projective topology. Then $\mathcal{A}_\infty$ becomes a topological space.
Basic properties of the test functionals

- If $f \in \mathcal{A}_\infty$, then, for $h_1, \ldots, h_n \in \mathcal{S}$ and for $p \in \mathbb{N}$,

  $$|D^n f(z) h_1 \cdots h_n| \leq \|f\|_{\mathcal{A}_p} \exp \left[\|z\|_{-p}^2 \left(\sum_{j=1}^{\infty} \|h_j\|_{-p}\right)^2\right].$$
Basic properties of the test functionals

- If \( f \in \mathcal{A}_\infty \), then, for \( h_1, \ldots, h_n \in \mathcal{S} \) and for \( p \in \mathbb{N} \),

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\]

- \( \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0)z^n \) converges to \( f \) in \( \mathcal{A}_\infty \) for any \( f \in \mathcal{A}_\infty \).
Basic properties of the test functionals

- If \( f \in A_\infty \), then, for \( h_1, \ldots, h_n \in S \) and for \( p \in \mathbb{N} \),

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\]

- \( \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0) z^n \) converges to \( f \) in \( A_\infty \) for any \( f \in A_\infty \).

- \( A_\infty \) is an algebra.
Basic properties of the test functionals

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$$\left| D^n f(z) h_1 \cdots h_n \right| \leq \|f\|_{\mathcal{A}_p} \exp \left[ \|z\|_p^2 \left( \sum_{j=1}^{\infty} \|h_j\|_{-p} \right)^2 \right].$$

- \( \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0) z^n \) converges to $f$ in $\mathcal{A}_\infty$ for any $f \in \mathcal{A}_\infty$.

- $\mathcal{A}_\infty$ is an algebra.

- The Wiener–Ito decomposition of $f \in \mathcal{A}_\infty$ converges to $f$ in $\mathcal{A}_\infty$. 

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Basic properties of the test functionals

- If $f \in A_\infty$, then, for $h_1, \ldots, h_n \in S$ and for $p \in \mathbb{N}$,

  $$|D^n f(z) h_1 \cdots h_n| \leq \|f\|_{A_p} \exp \left[ \|z\|^2_p \left( \sum_{j=1}^{\infty} \|h_j\|_{-p} \right)^2 \right].$$

- $$\sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0) z^n$$ converges to $f$ in $A_\infty$ for any $f \in A_\infty$.

- $A_\infty$ is an algebra.

- The Wiener–Ito decomposition of $f \in A_\infty$ converges to $f$ in $A_\infty$.

- For $f \in A_\infty$, define $F_{\alpha,\beta} f(y) = \int_{S^*} f(\alpha x + \beta y) \mu(dx)$ for $\alpha, \beta \in \mathbb{C}$. Then $F_{\alpha,\beta}(A_\infty) \subset A_\infty$ and $F_{\alpha,\beta}$ is continuous on $A_\infty$. 

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Topological Equivalence of \((S)\) and \(A_\infty[12]\)

Let \(f \in (S)\) and \(\tilde{f}\) be its analytic version of \(f\). Let \(p > \frac{1}{2}\) and \(r > \frac{1}{2}\). There exist some constants \(\alpha_p\) and \(\beta_p\) such that

\[
\alpha_p \|\tilde{\varphi}\|_{A_{p-1}} \leq \|\varphi\|_{2,p} \leq \beta_p \|\tilde{\varphi}\|_{A_{p+r}}. \tag{3.1}
\]
Given $F \in (S)^*$, recall that the $S$–transform of $F$ is defined as follows:

$$SF(\xi) = \begin{cases} 
\mu \ast F(\xi), & \text{if } F \in L^2[S', \mu]; \\
e^{-\frac{1}{2}|\xi|^2} \langle \langle F, e^{(\cdot, \xi)} \rangle \rangle, & \text{if } F \in (S)^*,
\end{cases}$$

where $\xi \in S$.

$SF$ is also denoted by $U_F$, $U_F$ is called the $U$–functional $F$. 

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Locality

In [12], it has been shown that, for any real number \( p \),

\[
\| F \|_{2,p}^2 \leq \sum_{n=0}^{\infty} \frac{1}{n!} \| D^n U_F(0) \|_{HS^n[S-P]}^2
\]

\[
= \lim_{n \to \infty} \int_{S'} \int_{S'} U_F(A^p P_n x + iA^p P_n y) \mu(dy) \mu(dx),
\]

where \( P_n \)'s are orthogonal projections of \( H \) which tend to the identity \( I_H \).

Proposition

Let \( p \in \mathbb{R}^1 \) and \( r > \frac{1}{2} \). Then we have

\[
\| F \|_{2,p} \leq C_r \| U_F \|_{A_{p+r}}.
\]
Browanian motion as a functional in $L^2(S')$

Let $(H, B)$ be an abstract Wiener space with abstract Wiener measure $\mu = p_1$. Let $B^*$ be the dual space which is regarded as the subspace of $H$.

- Let $\xi \in B^*$. Define
  \[ \tilde{\xi}(x) = (x, \xi). \]

Then $\tilde{\xi} \in A_\infty$ and $\tilde{\xi}$ is normal distributed with mean zero and variance $\|\xi\|_H^2$.
Browanian motion as a functional in $L^2(S')$

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- Let $\xi \in B^*$. Define
  $$\tilde{\xi}(x) = (x, \xi).$$

Then $\tilde{\xi} \in A_\infty$ and $\tilde{\xi}$ is normal distributed with mean zero and variance $\|\xi\|^2_H$.

- For any $h \in H$, there exists a sequence $(\xi_n) \subset B^*$ such that $|\xi_n - h|_H \to 0$. It follows that $\int_B |\tilde{\xi}_n - \tilde{\xi}_m|^2 \mu(dx) = \|\xi_n - \xi_m\|^2_H \to 0$ as $n, m \to \infty$. Thus $\{\tilde{\xi}_n\}$ forms a Cauchy sequence in $L^2(B)$ so that the $L^2(B)$-limit of $\{\xi^n\}$ exists. Define
  $$\tilde{h} = L^2(B) - \lim_{n \to \infty} \tilde{\xi}.$$

Then $\tilde{h} \sim N(0, \|h\|^2_H)$. In notation, we also write
  $$\tilde{h}(x) = \langle x, h \rangle.$$
The Brownian motion as a functional in $S'$, cont.

When $H = L^2(\mathbb{R})$, we consider $(L^2(\mathbb{R}), S')$ as the union of the abstract Wiener spaces $(L^2(\mathbb{R}), S_p)$. Then $h$ is well-defined as a normal distributed random variable with mean 0 and variance $|h|^2_0$.

- The Brownian motion on the probability space $(S', \mathcal{B}(S), \mu)$ may be represented by $B(t)$ defined by

$$B(t, x) = \begin{cases} 
\langle x, 1_{(0, t]} \rangle, & t \geq 0 \\
-\langle x, 1_{(t, 0]} \rangle, & t < 0,
\end{cases}$$

for almost all $x \in S'$. 

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Workshop on IDAQP and their Applications 3
The Brownian motion as a functional in $S'$, cont.

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-\langle x, 1_{(t,0]} \rangle, & t < 0,
\end{cases}$$

for almost all $x \in S'$.

- Let

$$h_t = \begin{cases} 
1_{(0,t]}, & t \geq 0 \\
-1_{(t,0]}, & t < 0,
\end{cases}$$

then

$$B(t, x) = \langle x, h_t \rangle.$$
White noise as a GWF

For any test functional $\varphi$, we have

$$\langle \dot{B}(t), \varphi \rangle = \frac{d}{dt} \langle B(t), \varphi \rangle$$

$$= \lim_{\epsilon \to 0} \int_{S^1} \frac{1}{\epsilon} \langle x, h_{t+\epsilon} - h_t \rangle \varphi(x) \mu(dx)$$

$$= \lim_{\epsilon \to 0} D\mu \varphi(0) \left\{ \frac{1}{\epsilon} (h_{t+\epsilon} - h_t) \right\}$$

$$= D\mu \varphi(0) \delta_t.$$

It is easy to see that the mapping $\varphi \to D\mu \varphi(0) \delta_t$ is continuous on $A_\infty$. This leads to the definition of white noise given as follows

$$\langle \dot{B}(t), \varphi \rangle = D\mu \varphi(0) \delta_t.$$
Composition of generalized function with random vectors

For any \( f \in S'(\mathbb{R}^n) \), and \( h_i \in L^2, \ i = 1, 2, 3 \ldots \), we may define \( f(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) \) formally by

\[
f(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \frac{1}{2\pi^n} \int_{\mathbb{R}^n} \hat{f}(u_1, \ldots, u_n) e^{i\sum_{j=1}^n u_i \tilde{h}_i} \, du_1 \ldots \, du_n.
\]

Then for \( \varphi \in A_\infty \), we have

\[
\langle\langle f(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n), \varphi \rangle\rangle = (f, \hat{G}_h, \varphi),
\]

\[
G_h,\varphi(u) = (1/\sqrt{2\pi})^n F_{1, i\varphi([u, h])} \exp\left(-\frac{1}{2} \|[u, h]\|_0^2 \right),
\]

where \( u = (u_1, u_2, \ldots, u_n) \), \( h = (h_1, h_2, \ldots, h_n) \) and \([u, h] = \sum_{j=1}^n u_i h_i\).
The Donsker delta function $\delta_x(B(t))(t > 0)$ may be defined by

$$\langle \delta_x(h_t), \varphi \rangle := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ias-\frac{1}{2}s^2t} F_{1,i} \varphi(sh_t) ds$$

$$:= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ias-\frac{1}{2}s^2t} \left\{ \int_{S'} \varphi(y + isht) \mu(dy) \right\} ds.$$
Itô’s formula

For \( f \in S' \), define \( f(B(t)) = f(\tilde{h}_t) \) for \( t > 0 \) by

\[
\langle \langle f(B(t)), \varphi \rangle \rangle := (f, \hat{G}_t, \varphi)
\]

where \( \hat{G}_{t, \varphi}(u) = \left(1/\sqrt{2\pi}\right)\mathcal{F}_{1,i\varphi(u1_{(0,t]})} \exp\left(-\frac{1}{2}u^2t\right) \).

If we differentiate \( f(B(t)) \) with respect to \( t \) we immediately obtain:

\[
\frac{d}{dt} \langle \langle f(B(t)), \varphi \rangle \rangle = (f_{[u]}, iu \left\{ 1/\sqrt{2\pi}\mathcal{F}_{1,i\partial_t\varphi(u1_{(0,t]})} e^{-\frac{1}{2}u^2t} \right\})
\]

\[
+ (f_{[u]}, -\frac{1}{2}u^2 \left\{ (1/\sqrt{2\pi})\mathcal{F}_{1,i\partial_t\varphi(u1_{(0,t]})} e^{-\frac{1}{2}u^2t} \right\})
\]

\[
= \langle \langle f(B(t)), \partial_t\varphi \rangle \rangle + \langle \langle \frac{1}{2}f''(B(t)), \varphi \rangle \rangle.
\]
The generalized Itô’s formula follows:

\[
\frac{d}{dt} f(B(t)) = \partial_t^* f'(B(t)) + \frac{1}{2} f''(B(t)).
\]

It can be shown that

\[
\int_a^b \partial_t^* f'(B(t)) dt = \int_a^b f'(B(t)) dB(t).
\]

If one replace the Brownian motion by any other normal processes

\[X_t(x) = \langle x, \beta_t \rangle,\]

one may derive a new Itô formula by differentiating \( f(X_t) \) with respect to \( t \).
Hitsuda Formula (cf. Kuo [10])

Define

\[ \langle f(B(t), B(1), \varphi) \rangle = (f, \hat{H}_{t,\varphi}), \quad \text{with} \]

\[ \hat{H}_{t,\varphi} = \frac{1}{\sqrt{2\pi}} \mathcal{F}_{1,i\varphi}(uh_t + \nu h_1)e^{-\frac{1}{2}u^2\|uh_t + \nu h_1\|^2}. \]

Differentiating with respect to \( t \) and then integrating from \( a > 0 \) to \( b > a(1 > b) \) we obtain

\[ f(B(b), B(1)) - f(B(a), B(1)) = \int_a^b \partial_t^* f_x(B(t), B(1)) \, dt \]

\[ + \int_a^b f_{xy}(B(t), B(1)) \, dt + \frac{1}{2} \int_a^b f_{xx}(B(t), B(1)) \, dt. \]
An application of Hitsuda formula

Apply Hitsuda formula with $f(xy) = xy$, we immediately have

$$B(b)B(1) - B(a)B(1) = \int_a^b \partial_t^* B(1) \, dt + (b - a),$$

or,

$$\int_a^b \partial_t^* B(1) \, dt = (B(b) - B(a))B(1) - (b - a).$$
Itô formula for non-adapted Processes

For the more general case $f(X_t)$ with $X_t(x) = \langle x, h_t \rangle$ with $\{X_t\}$ being a normal processes (which is non-adapted generally), one may also apply the same argument above to derive the following “Itô” formula:

$$f(X(b)) = f(X(a)) + \int_a^b D^*_{h_t} f'(B(t)) \, dt + \int_a^b \left\{ \frac{d}{dt} \| h_t \|_0 \right\} f''(X(t)) \, dt,$$

where $\dot{h}_t = \frac{d}{dt} h_t$.

Again a new integral such as $\int_a^b D^*_{h_t} f(t) \, dt$ arises.
Itô formula for Brownian Bridge

The Brownian Bridge $X(t)$ may represented by

\[ X(t) = B(t) - tB(1) = \tilde{\beta}_t = \tilde{h}_t - t\tilde{h}_1, \quad (\beta_t = h_t - th_1). \]

Clearly $\|\beta_t\|_0^2 = t - t^2$. Let $k_t = \frac{d}{dt}\beta_t$. Then, for $f \in S'$, we have

\[ f(X(b)) - f(X(a)) = \int_a^b D^*_k f'(X(t)) \, dt + \int_a^b \frac{1}{2} (1 - 2t)f''(X(t)) \, dt \]

which exist in the generalized sense, where $0 < a < b < 1$. 
Let \( \{ Y_t : a \leq t \leq b \} \), \( 0 < a < b < 1 \) be a continuous \((S)^*\)-valued process, we define
\[
\int_a^b Y_t \, dX(t+) := \lim_{|\Gamma| \to 0} \sum_{j=1}^n (\widetilde{\beta}_{t_j} - \widetilde{\beta}_{t_{j-1}}) Y_{t_{j-1}}
\]
provided that the limit exist in \((S)^*\), where
\( \Gamma = \{ a = t_0 < t_1 < \cdots < t_n = b \} \)

Then one can show that
\[
\int_a^b D^*_k f'(X(t)) \, dt = \int_a^b f'(X(t)) \, dX(t+) + \int_a^b t \, f''(X(t)) \, dt.
\]

The above identity also give the probabilistic meaning of the stochastic integral
\[
\int_a^b D^*_k f'(X(t)) \, dt
\]
Kuo’s stochastic integral

In [W. Ayed and H.H. Kuo: An extension of the Itô formula, v.2, COSA(2008),323-333], the authors define the following stochastic integral: Let \( f(t), \ a \leq t \leq b, \) be adapted and \( \varphi(t), \ a \leq t \leq b, \) be instantly independent (i.e. \( \varphi(t) \) is independent of \( \sigma\{B(s), s \leq t\} \)). Define the stochastic integral of \( f(t)\varphi(t) \) by

\[
I(f \varphi) = \int_a^b f(t)\varphi(t) \, dB(t) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^n f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1}))
\]

provided the limit exist in probability.

The main results showed that

\[
\int_a^b f(t)\varphi(t) \, dB(t) = \int_a^b \partial^*[f(t)\varphi(t)] \, dB(t).
\]
Composition of tempered distribution with Lévy process

Assumption: \( \sigma^2 = \beta(0) - \beta(0-) > 0. \)

Let \( \Lambda \) be the Lévy probability measure. Applying the inversion formula of Fourier transform, we have for \( f \in \mathcal{S}, \)

\[
f(X(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(r) e^{irX(t)} \, dr,
\]

where \( \mathcal{F}f \) is the Fourier transform of \( f \). Then, for any test functionals \( \varphi \), we have

\[
\langle \langle f(X(t)), \varphi \rangle \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(r) \left\{ \int_{S'} \varphi(x) e^{irX(t;x)} \Lambda(dx) \right\} \, dr.
\]

(5.1)
Let $G_{t,\varphi}$ be the map from $\mathbb{R}$ to $\mathbb{C}$ defined by

$$G_{t,\varphi}(r) = \frac{1}{\sqrt{2\pi}} \int_{S'} \varphi(x) e^{irX(t;x)} \Lambda(dx).$$

Recall the relation between $\mathcal{T}$-transform and $S$-transform

$$\mathcal{T} \varphi(\eta) = \mathbb{E}[e^{i\langle \cdot, \eta \rangle}] S\varphi(\phi_{i\eta}). \quad (5.2)$$

for $\eta \in L^1 \cap L^2(\mathbb{R}, dt)$, where $\phi_{\xi}(t, u) = (e^{\xi^*(t,u)} - 1)/u$ if $u \neq 0$; otherwise, $\phi_{\xi}(t, u) = \xi(t, u)$.

It follows from the identity (5.2) that we obtain

$$G_{t,\varphi}(r) = \frac{\mathbb{E}[e^{irX(t)}]}{\sqrt{2\pi}} \times S\varphi(\phi(t, r)),$$

where $\phi : (0, +\infty) \times \mathbb{R} \to L^2_c(\mathbb{R}^2, \lambda)$ is given by

$\phi(t, r)(s, u) = (e^{iru1_{[0,t]}(s)} - 1)/u$, if $u \neq 0$; otherwise, $\phi(t, r)(s, u) = ir1_{[0,t]}(s)$. 
Lemma[16]

For a fixed \( \varphi \in \mathcal{L} \), \( G_{t,\varphi} \) is a function in \( S_c \). In fact, for any \( q \geq 0 \) and \( 0 < a < b < +\infty \), there exists a positive real number \( p \), depending only on \( q, a, b \), such that

\[
|G_{t,\varphi}|_q \leq \|\varphi\|_p
\]

uniformly in \( t \) on the compact interval \([a, b]\).
The above lemma implies that the mapping $\varphi \in \mathcal{L} \mapsto G_{t,\varphi}$ is a continuous $S_c$-valued map. It follows that the composition $F(X(t))$ is well-defined for $F \in S'$ in the following

**Definition**

For $F \in S'$ and $t > 0$, we define $\langle \langle F(X(t)), \varphi \rangle \rangle = \langle \mathcal{F} F, G_{t,\varphi} \rangle$ for $\varphi \in \mathcal{L}$, where $\mathcal{F} F$ is the Fourier transform of $F$ and $(\cdot, \cdot)$ is the $S'_c$-$S_c$ pairing. In particular, when $F = \delta_a$, the Dirac delta function concentrated on the point $a$, $\delta_a(X(t)) (= \delta(X(t) - a))$ is called the Donsker’s delta function of the Lévy process $X$. 
Note that
\[ \delta(X(t) - a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ir(X(t) - a)} \, dr \quad \text{in } \mathcal{L}_{-p,c}, \]
for any sufficiently large \( p > 0 \) so that \( \sigma^2 - (\tau_2/\lambda_0^2p) > 0 \), where the integral exists in the sense of Bochner (see [16]). Moreover,
\[ \langle \langle F(X(t) - a), \varphi \rangle \rangle = (F[s], \langle \langle \delta(X(t) - a - s), \varphi \rangle \rangle), \]
where \((\cdot, \cdot)\) is the \(S'\)-\(S\) pairing, and \(F[s]\) means that \(F\) acts on the test functions in the variable \(s\).
Itô formula for $F(X(t))$[17]

We are ready to show the Itô formula for the $\mathcal{L}'$-valued process $F(X(t))$ with $F \in S'$, $t > 0$.

By differentiating $F(X(t))$ with respect to $t$, we obtain Let $F \in S'$. Then, for $b > a > 0$,

$$F(X(b)) - F(X(a)) = \tau_1 \int_a^b F'(X(t)) \, dt$$

$$+ \int_a^b \int_{-\infty}^{+\infty} \frac{\kappa_u F(X(t)) - F(X(t)) - u F'(X(t))}{u^2} \, d\lambda(t, u)$$

$$+ \int_a^b \int_{-\infty}^{+\infty} \partial^*_{(t,u)} \frac{\kappa_u F(X(t)) - F(X(t))}{u} \, d\lambda(t, u),$$

in $\mathcal{L}'$, where $F'$ is the first distribution derivative of $F$, $\kappa_u F = F(\cdot + u)$ is the translate of $F$; and the integrals exist in the sense of Bochner.
Representation of Complex Brownian Motion

• By a complex Brownian functionals we mean a function of complex Brownian motion given by

\[ Z(t, \omega) = B_1(t, \omega) + iB_2(t, \omega) \]

where \( B_1 \) and \( B_2 \) are independent real-valued standard Brownian motions. \( Z(t) \) is normally distributed with mean zero and variance parameter \( |t| \).
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where $\mu_t$ denotes the Gaussian measure defined on $S'$ with characteristic function given by

$$C(\xi) = \int_{S'} e^{i(x,\xi)} \mu_t(dx) = e^{-t|\xi|^2/2}.$$

- The complex Brownian motion on $(S'_c, \mathcal{B}(S'_c), \nu(dz))$ may be represented by

$$Z_t(x,y) = \langle x, h_t \rangle + i \langle y, h_t \rangle,$$

where

$$h_t = \begin{cases} 
1 & t \geq 0, \\
-1 & t < 0.
\end{cases}$$
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\]

where

\[
h_t = \begin{cases} 
1_{(0,t]} & , \ t > 0, \\
-1_{[t,0]} & , \ t < 0.
\end{cases}
\]
The calculus of complex Brownian functional is then performed with respect to the measure $\mu(dz)$. 

For example, let $f: \mathbb{C} \to \mathbb{C}$ be an entire function of exponential growth. Then we have 

$$E[|f(Z(t))|^2] = \int S' \int S' |f(\langle x + iy, h \rangle)|^2 \mu^{1/2}(dx) \mu^{1/2}(dy).$$ 

The above identity gives a connection between the function of complex Brownian motion and the Segal-Bargmann entire functionals.
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The above identity gives a connection between the function of complex Brownian motion and the Segal-Bargmann entire functionals.
Itô formula for entire Brownian Functionals

We shall show that, for any Segal-Bargmann entire function $F$, the Itô formula is given by

$$ F(Z(b)) - F(Z(a)) = \int_a^b F'(Z(t))dZ(t). $$
Definition of Segal-Bargmann space $L[12]$

A single-valued function $f$ defined on $H_c$ is called a Segal-Bargmann entire function if it satisfies the following conditions:

(i) $f$ is analytic in $H_c$.

(ii) The number

$$M_f := \sup_P \int_H \int_H |f(Px + iPy)|^2 n_t(dx)n_t(dy)$$

is finite, where $n_t$ denoted as the Gaussian cylinder measure on $H$ with variance parameter $t > 0$ and $P$’s run through all orthogonal projections on $H$. 
Denote the class of Segal-Bargmann entire function on $H$ by $SB_t[H]$ and define $\|f\|_{SB_t[H]} = \sqrt{M_f}$. Then $(SB_t[H], \|\cdot\|_{SB_t[H]})$ is a Hilbert space.
It follows immediately from L[12] that we have

\[ \| f \|_{SB_t[H]}^2 = \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \left( \sum_{i_1, \ldots, i_k=1}^{N} |D^k f(0)e_{i_1} \cdots e_{i_k}|^2 \right) \]

\[ = \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \| D^n f(0) \|_{HS^2[H]}^2, \quad (6.1) \]

When \( t = 1/2 \), we simply denote \( SB_t[H] \) by \( SB[H] \).
where $\|S\|_{HS^n[H]}$ denotes the Hilbert-Schmidt norm of a n-linear operator $S \in L^n(H)$ defined by

$$
\|S\|_{HS^n(H)} := \left( \sum_{i_1,\ldots,i_k=1}^{\infty} |Se_{i_1} \cdots e_{i_k}|^2 \right)^{1/2}
$$

which is independent of the choice of CONS $\{e_i\}$ of $H$. 

Definition of Infinite-dimensional Segal-Bargman entire functionals

Definition
For each $p \in \mathbb{R}$, define

$$\| \phi \|_p = \left( \sum_{n=0}^{\infty} \frac{\| D^n \phi(0) \|_{HS^n[S_{-p}]}^2}{n!} \right)^{1/2}$$

and set

$$SB_p = \{ \phi \in SB[S_{-p}] : \| \phi \|_p < \infty \}$$
Let $SB_\infty$ be the projective limit of $SB_p$ for $p \geq 0$ and let $SB'_\infty$ be the dual space of $SB_\infty$. We note that

$$SB_\infty = A_\infty.$$ 

$SB_\infty$ is a nuclear space and we have the following continuous inclusions:

$$SB_\infty \subset SB_p \subset SB[L^2] = SB \subset SB'_p \subset SB'_\infty.$$
The space $SB_\infty$ will serve as test functionals and $SB'_\infty$ is referred as the generalized complex Brownian functionals.
The space $SB'_p$ may be identified as the space of entire functions defined on $S_{p,c}$ such that: $\|\phi\|_p < \infty$ and the pairing of $SB'_\infty$ and $SB_\infty$ is defined by

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle\langle D^n \Phi(0), D^n \varphi(0) \rangle\rangle_{HS^n},$$

where

$$\langle\langle D^n \Phi(0), D^n \varphi(0) \rangle\rangle_{HS^n} := \sum_{i_1, \ldots, i_n=1}^{n} \left[ D^n \Phi(0) e_{i_1} \cdots e_{i_n} D^n \varphi(0) e_{i_1} \cdots e_{i_n} \right].$$
One-dimensional Segal-Bargman entire functions

If $\phi(z)$ can be represented by a formal power series $\sum_{n=0}^{\infty} a_n z^n$, we define

$$\| \phi \|_p = \left( \sum_{n=0}^{\infty} (2n + 2)^{2p} n! |a_n|^2 \right)^{1/2}$$

and let

$$SB_p(\mathbb{R}) = \{ \phi : \| \phi \|_p < \infty \}$$
If \( \phi(z) \) is a formal power series represented by \( \sum_{n=0}^{\infty} b_n z^n \), we define

\[
|\phi|_{-p} = \left( \sum_{n=0}^{\infty} n! |b_n|^2 (2n + 2)^{-2p} \right)^{1/2}.
\]

Then the dual space \( SB'_{p} \) of \( SB_{p} \) is characterized by

\[
SB_{-p}(\mathbb{R}) = \{ \phi : |\phi|_{-p} < \infty \}
\]

The space \( SB_{\infty}[\mathbb{R}] \) is defined as the projective limit of \( SB_{p}[\mathbb{R}] \) with dual space \( SB'_{\infty}[\mathbb{R}] = \bigcup_{p>0} SB'_{p}[\mathbb{R}] \).
Composition of generalized function with complex Brownian motion

Let $\psi \in SB_{\infty}$. Then, for any one dimensional generalized Segal-Bargman entire function $f \in SB'_{\infty}(\mathbb{R})$, represented by $\psi(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$\langle\langle f(Z(t)), \psi \rangle\rangle_c = \sum_{n=0}^{\infty} b_n D^n \psi(0) h_t^n.$$  \hspace{1cm} (6.2)

(6.2) gives the definition of $f(Z(t))$. 

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It can be proved that the composition $f(Z(t))$ defined in Example 6 is in fact a generalized Segal-Bargmann functional. This follows straight forward from the identity (6.2). We state it as a theorem without proof.

**Theorem**

Let $\psi \in SB_\infty$. If $f \in SB'_\infty(\mathbb{R})$, then $f(Z(t))$, defined by (6.2), is a member of $SB'_\infty$. More precisely, for each $t > 0$, $\exists p \ni |h_t|_{-p} \leq 1$, and

$$\langle\langle f(Z(t)), \psi \rangle\rangle_c \leq \|f\|_{\alpha-1}\|\psi\|_{\alpha,p}$$
Itô formula for Generalized Complex Brownian functionals

Let $f \in SB'_\infty(\mathbb{R})$. Then we have

$$\frac{d}{dt}\langle\langle f(Z(t)), \phi \rangle\rangle_c = \sum_{n=0}^{\infty} b_n D_n^\phi(0) h_t^{n-1} \delta_t = \sum_{n=0}^{\infty} b_n nD^{n-1}(D\phi(0)\delta_t)h_t^{n-1}$$

$$= \sum_{n=0}^{\infty} b_n nD^{n-1}(\partial_t\phi)(0) h_t^{n-1} = \langle\langle \partial^*_t f'(Z(t)), \phi \rangle\rangle_c$$

where $\partial_t = \partial_{\delta_t}$ and $\partial^*_t$ is the adjoint operator of $\partial_t$. It follows that

$$\frac{d}{dt} f(Z(t)) = \partial^*_t f'(Z(t)).$$
This proves the Itô formula for complex Brownian motion. As a summary, we state the above result as a theorem.

**Theorem**

Let \( f \in \mathcal{SB}'_{\infty}(\mathbb{R}) \). Then we have

\[
\frac{d}{dt} f(Z(t)) = \partial_t^* f'(Z(t)).
\]

or in the integral form,

\[
f(Z(b)) - f(Z(a)) = \int_a^b \partial_t^* f'(Z(t)) dt.
\]
As in the case of real Brownian motion, the term on the right hand side of Itô formula may be interpreted as stochastic integral as shown below.

**Definition**

Suppose that $f \in \mathcal{SB}'_{\infty}$. Define the stochastic integral $f(Z(t))$ as follows:

$$
\langle\langle \int_{a}^{b} f(Z(t))dZ(t), \phi \rangle\rangle_{c} = \lim_{\|\triangle n\| \to 0} \langle\langle \sum_{i=1}^{n} f(Z(t_{i-1}))(Z(t_{i}) - Z(t_{i-1})), \phi \rangle\rangle_{c}
$$

where $a = t_{0} < t_{1} < t_{2} < \cdots < t_{n} = b$ and $\|\triangle n\| = \max_{j} |t_{j} - t_{j-1}|$. 
**Theorem**

Let $f \in SB_\alpha(\mathbb{R})$ and $\phi \in SB_\alpha$. Then

$$
\langle \int_a^b f(Z(t))dZ(t), \phi \rangle_c = \langle \int_a^b \partial_t^* f(Z(t))dt, \phi \rangle_c.
$$
A connection between the Itô formulas for the complex and real Brownian motion

Recall the Itô formula of \( f(t, B_t) \),

\[
f(b, B_b) - f(a, B_a) = \int_a^b f_t(t, B_t) \, ds + \int_a^b f_x(t, B_t) \, dB_t \\
+ \frac{1}{2} \int_a^b f_{xx}(t, B_t) \, dt.
\]

Take \( S- \)transform, we obtain

\[
\mu_b f(\langle \xi, h_b \rangle) - \mu_a f(\langle \xi, h_a \rangle) = \int_a^b \xi(t)(\mu_t f)'(\langle \xi, h_t \rangle) \, dt + \frac{1}{2} \mu_t f''(\langle \xi, h_t \rangle) \, dt,
\]

where \( \mu_t f(u) = \int_{\mathbb{R}} f(u + \sqrt{t}v) \mu(\,dv) \).
Replace $\xi$ by $\dot{Z}(t)$ in the above equation, we have

$$\mu_b f(Z_b) - \mu_a(Z_a) = \int_a^b \mu_t f'(Z_t) dZ_t + \frac{1}{2} \int_a^b \mu f''(Z_t) dt.$$ 

The above formula indeed follows from the Itô formula of complex Brownian motion by applying the Itô formula to $\mu_t f(t, Z_t)$:

$$\mu_b f(Z_b) - \mu_a(Z_a) = \int_a^b \mu_t f'(Z_t) dZ_t + \int_a^b \frac{d}{dt} (\mu_t f)(Z_t) dt,$$

where the last term is verified by the following computation

$$\int_a^b \frac{d}{dt} (\mu_t f)(Z_t) dt = \frac{1}{2} \int_a^b \frac{1}{\sqrt{t}} \int_{\mathbb{R}} [f'(Z_t + \sqrt{tu})] \cdot u \mu(du) dt$$

$$= \frac{1}{2} \int_a^b \mu_t f''(Z_t) dt.$$
Example

To evaluate the integral

$$I = \int_a^b \partial_t^* B(1) \, dt.$$  

We first take $S$-transform of $I$ to obtain

$$S(I)(\xi) = e^{-\frac{1}{2} \|\xi\|^2} \int_a^b \left\langle \left\langle \partial_t^* B(1), e^{\left\langle \cdot, \xi \right\rangle} \right\rangle \right\rangle \, dt = \int_a^b \xi(t)\langle \xi, h_1 \rangle \, dt.$$  

Replace $\xi$ by $\dot{Z}$, we obtain

$$S(I)(\dot{Z}) = \int_a^b \dot{Z}(t)\langle \dot{Z}, h_1 \rangle \, dt = (Z(b) - Z(a))Z(1).$$  

It follows that

$$I = \int_{\mathcal{S}'} \left\langle x + iy, h_b - h_a \right\rangle \left\langle x + iy, h_1 \right\rangle \mu(dy) = B(1)(B(b) - B(a)) - (b - a).$$
A remark

The above theory remains true that if we replace the Brownian motion and the associated Hilbert space $H = L^2(\mathbb{R}, dx)$ by a Lévy process together with the Hilbert space $L^2(\mathbb{R}^2, d\lambda)$. 
Volterra Gaussian processes [1]

Consider the Gaussian Processes of the form

$$Z(t) = \int_0^t K(t, s) dB(s), \quad 0 \leq t \leq 1,$$

where $K(t, s)$ is a kernel function from $[0, 1] \times [0, 1]$ into $\mathbb{R}$ satisfying

$$\sup_{t \in [0,1]} \int_0^t |K(t, s)|^2 \, ds < +\infty.$$

For each $t \in [0, 1]$, define

$$K_t(s) = \int_0^{s \wedge t} K(t, u) \, du, \quad 0 \leq s \leq 1.$$

Then $K_t \in \mathcal{H}$ with $\dot{K}_t = K(t, \cdot) \cdot 1_{[0,t]}$. 
Moreover,
\[ Z(t) = \langle \cdot, K_t \rangle, \quad 0 \leq t \leq 1, \quad \text{on } (C, \mathcal{B}(C), \omega). \]

Let \( F \in L^1(\mathbb{R}) \) and \( \varphi \) be a test Brownian functional. Then
\[
\langle F(Z(t)), \varphi \rangle = \int_C F(\langle x, K_t \rangle) \varphi(x) \omega(dx)
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_C \left\{ \int_{-\infty}^{\infty} \hat{F}(u) e^{i\langle x, K_t \rangle u} du \right\} \varphi(x) \omega(dx)
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(u) \left\{ \int_C \varphi(x + i u K_t) \omega(dx) \right\} e^{-\frac{1}{2} u^2 \int_0^t |K(t,s)|^2 ds} du
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(u) \cdot S \varphi(i u K_t) \cdot e^{-\frac{1}{2} u^2 \int_0^t |K(t,s)|^2 ds} du,
\]
where \( \hat{F}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(r) e^{-i ur} dr \).
Define

\[ G_t, \varphi(u) = \frac{1}{\sqrt{2\pi}} S\varphi(i uK_t) \cdot e^{-\frac{1}{2}u^2 \int_0^t |K(t,s)|^2 \, ds}, \quad u \in \mathbb{R}. \]

Then \( G_t, \varphi \in S \). Thus, for \( F \in S' \), we can define \( F(Z(t)) \) as a generalized Brownian functional by

\[ \langle\langle F(Z(t)), \varphi \rangle\rangle = (\hat{F}, G_t, \varphi), \]

where \( (\cdot, \cdot) \) is the \( S'-S \) dual pairing.
Assume that $K(t, u)$ is differentiable in the variable $t$ in \{(t, u); 0 < u \leq t < 1\}, and both $K$ and $\frac{\partial K}{\partial t}$ are continuous in \{(t, u); 0 < u \leq t < 1\}. For $0 < t < 1$, let

$$h_t(s) = K(t, t) \cdot 1_{[t, 1]}(s) + \int_0^{s \wedge t} \frac{\partial K}{\partial t}(t, u) \, du, \quad s \in [0, 1].$$

Then

$$\lim_{\epsilon \to 0} \frac{K_{t+\epsilon} - K_t}{\epsilon} = h_t \quad \text{in } L^2([0, 1]).$$
There are two typical Volterra Gaussian processes as follows:

(a) (Fractional Brownian motion)

Let

\[ K(t, s) := K_H(t, s) = \]

\[ \begin{cases} 
  c_H \mathbf{1}_{[0, t]}(s) (t - s)^{H - \frac{1}{2}} \int_0^1 u^{H - \frac{3}{2}} \left(1 - \left(1 - \frac{t}{s}\right) u\right)^{H - \frac{1}{2}} \, du, & (H \in (\frac{1}{2}, 1)), \\
  \mathbf{1}_{[0, t]}(s), & (H = \frac{1}{2}), \\
  b_H \left[ \left(\frac{t}{s}\right)^{H - \frac{1}{2}} (t - s)^{H - \frac{1}{2}} \\
  - \left(H - \frac{1}{2}\right) s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{1}{2}} u^{H - \frac{3}{2}} \, du \right], & (H \in (0, \frac{1}{2})). 
\end{cases} \]
where

\[ c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}} \quad \text{and} \quad b_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}} \]

\((\beta(z, w) = \int_0^1 x^{z-1}(1-x)^{w-1} \, dx \text{ with } \Re z, \Re w > 0)\). Then
\{Z(t); \, t \in [0, T]\} is a fractional Brownian motion (fBm for short) of Hurst index \(H \in (0, 1)\) (see [2]).

(b) (Brownian bridge) Let \(T = 1\) and

\[ K(t, s) = \begin{cases} \frac{1-t}{1-s}, & \text{if } 0 \leq s \leq t < 1, \\ 0, & \text{otherwise.} \end{cases} \]


Thank You for your attention!