The theory of universally Baire sets in $2^{\omega_1}$

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Joint work with Matteo Viale

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We work in ZFC unless clearly specified.
Goal & Today’s topic

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Understand the theory of subsets of $\omega_1$ under “ZFC + Large Cardinals + Forcing Axioms” as much as the theory of subsets of $\omega$ (“reals”) under “ZFC + Large Cardinals”.

Today
Goal & Today’s topic

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Understand the theory of subsets of $\omega_1$ under “ZFC + Large Cardinals + Forcing Axioms” as much as the theory of subsets of $\omega$ (“reals”) under “ZFC + Large Cardinals”.

Today

Will generalize the notion of universally Baireness for subsets of $\mathcal{P}(\omega)$ (“sets of reals”) to that for subsets of $\mathcal{P}(\omega_1)$.
Motivation; Universally Baire sets of reals

- Large Cardinals
- Inner Model Theory
- UB sets
- Determinacy
- Generic Absoluteness
The size of the continuum is bigger than $\aleph_0$.

**Definition**

The **Continuum Hypothesis** states that the size of the continuum is $\aleph_1$.

**Theorem (Gödel)**

In ZFC, $L \models \text{"ZFC + CH"}$. In particular, one cannot refute CH in ZFC.
**Background; Continuum Hypothesis and Gödel**

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Gödel’s Program

Decide the truth-values of mathematically interesting statements independent of ZFC in “well-justified” extensions of ZFC.
Gödel’s Program
Decide the truth-values of mathematically interesting statements independent of ZFC in “well-justified” extensions of ZFC.

Theorem (Cohen)
Using forcing, one can show that one cannot prove CH in ZFC.

Theorem (Levy, Solovay)
Using forcing, one can show that large cardinals cannot decide the truth-value of CH in ZFC.
Background; Gödel’s Program and Large Cardinals

Theorem (Shelah, Woodin)
Assuming large cardinals, every set of reals in $L(\mathbb{R})$ is Lebesgue measurable.

Theorem (Woodin)
Assuming large cardinals, the 1st-order theory of $(L(\mathbb{R}), \in)$ is invariant under set generic extensions, i.e., for all 1st-order sentences $\phi$, $(L(\mathbb{R}), \in)^V \models \phi \iff (L(\mathbb{R}), \in)^{V_P} \models \phi$ for all partial orders $P$. 
Forcing Axioms are a generalization of Baire Category Theorem:

**Baire Category Theorem**

If $X$ is either a compact Hausdorff space or a completely metrizable space, then the intersection of countably many dense open sets is non-empty.
Background; Forcing Axioms

Forcing Axioms are a generalization of Baire Category Theorem:

**Baire Category Theorem**

If $X$ is either a compact Hausdorff space or a completely metrizable space, then the intersection of countably many dense open sets is non-empty.

**Convention**

From now on,

- $\kappa$ will always be an infinite cardinal, and
- $X$ will always be a topological space.

**Definition**

Let $BC_\kappa(X)$ state that the intersection of $\kappa$-many dense open sets in $X$ is non-empty.
### Convention
From now on, $B$ will always be a complete Boolean algebra.

### Definition
The **Stone space** of $B$ (denoted by $\text{St}(B)$) is the collection of ultrafilters on $B$ topologized by the sets $O_b = \{ G \in \text{St}(B) \mid b \in G \}$ for $b \in B$.

### Remark
Every Stone space $\text{St}(B)$ is a compact Hausdorff space.
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From now on, $B$ will always be a complete Boolean algebra.

Definition
The Stone space of $B$ (denoted by $\text{St}(B)$) is the collection of ultrafilters on $B$ topologized by the sets $O_b = \{ G \in \text{St}(B) \mid b \in G \}$ for $b \in B$.

Remark
Every Stone space $\text{St}(B)$ is a compact Hausdorff space.

Definition
The forcing axiom for $B$ at $\kappa$ (denoted by $\text{FA}_{\kappa}(B)$) states that $\mathcal{BC}_{\kappa}(X)$ holds for $X = \text{St}(B)$. 
Background; Forcing Axioms ctd..

Remark

1. For any compact Hausdorff $X$, there exists a $B$ such that

\[ BC_\kappa(X) \iff FA_\kappa(B). \]

2. Let $\kappa = \omega_1$. There is a $B$ such that $FA_{\omega_1}(B)$ fails. In fact, if $FA_{\omega_1}(B)$ holds, then $B$ must be stationary set preserving.
Remark

1. For any compact Hausdorff $X$, there exists a $B$ such that

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Definition

The axiom **Martin’s Maximum (MM)** states that $FA_{\omega_1}(B)$ holds for all $B$ which are stationary set preserving.

Note: The axiom $MM^{+++}$ is a technical strengthening of MM.
Theorem (Foreman, Magidor, and Shelah)

1. The theory ZFC + MM is consistent assuming the consistency of ZFC + Large Cardinals.
2. MM implies that the size of the continuum is $\mathfrak{c}_2$. 

Background: Forcing Axioms ctd...

Theorem (Foreman, Magidor, and Shelah)

1. The theory ZFC + MM is consistent assuming the consistency of ZFC + Large Cardinals.
2. MM implies that the size of the continuum is $\aleph_2$.

Theorem (Viale)

Assuming large cardinals and MM$^{+++}$, the following holds: For any $B$ which is stationary set preserving with $\models_B$ MM$^{+++}$,

$$(\omega_1, \mathcal{P}(\omega_1), \in)^V \prec^\Sigma_\omega (\omega_1, \mathcal{P}(\omega_1), \in)^{V^B}.$$ 

Furthermore,

$$(L(\text{Ord}^{\omega_1}), \in)^V \equiv (L(\text{Ord}^{\omega_1}), \in)^{V^B}.$$
We will identify \( \mathcal{P}(\kappa) \) with \( 2^\kappa = \{x \mid x : \kappa \to 2\} \).

We will consider \( 2^\kappa \) as the **product space** of the discrete space 2.
Universally Baire sets; Preparation

Convention

1. We will identify $\mathcal{P}(\kappa)$ with $2^\kappa = \{x \mid x : \kappa \to 2\}$.
2. We will consider $2^\kappa$ as the **product space** of the discrete space 2.

Why consider the product space on $2^\kappa$?

Want to generalize the correspondence $(f \mapsto \tau_f, \tau \mapsto f_\tau)$ between continuous functions from $St(B)$ to $2^\omega$ and $B$-names for a subset of $\omega$ with the following properties:

1. $f(\tau_f) = f$ and $\forces_B \tau(f_\tau) = \tau$, and
2. for any $Z \prec_{\Sigma_{2014}} V$ with $B, f, \tau \in Z$, and any $(Z, B)$-generic $g$, $\text{val}_g(\tau_f) = f(g)$ and $f_\tau(g) = \text{val}_g(\tau)$. 

We will identify $\mathcal{P}(\kappa)$ with $2^\kappa = \{ x | x: \kappa \to 2 \}$.

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### Observation

Using the product topology on $2^\kappa$, one can generalize the above correspondence to the one in the context of $2^\kappa$. 

Universally Baire sets; Preparation ctd.

Definition

Let $A \subseteq X$.

1. The set $A$ is **nowhere dense** if $A$ is disjoint from an open dense subset of $X$.

2. The set $A$ is **$\kappa$-meager** if it is the union of $\kappa$-many nowhere dense sets in $X$. 
Definition

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Baire Category Theorem, reformulated

Let $\kappa = \omega$. If $X$ is either compact Hausdorff or completely metrizable, then $X$ is NOT $\omega$-meager.

Remark

The space $X$ is $\kappa$-meager if and only if $\text{BC}_{\kappa}(X)$ fails.
Universally Baire sets; Preparation ctd..

**Definition**

Let $A \subseteq X$. The set $A$ has the $\kappa$-Baire property in $X$ if there is an open set $U$ in $X$ such that $U \triangle A = (U \setminus A) \cup (A \setminus U)$ is $\kappa$-meager.

**Remark**

1. The collection of subsets of $X$ with the $\kappa$-Baire property is closed under complements and unions of $\kappa$-many sets, and it contains all the open sets in $X$.
2. If $X$ is $\kappa$-meager, then every subset of $X$ has the $\kappa$-Baire property.
Universally Baire sets; The case for subsets of $2^\omega$

**Definition (Feng, Magidor, and Woodin)**
Let $A \subseteq 2^\omega$. We say $A$ is universally Baire if for any compact Hausdorff $X$ and any continuous $f : X \to 2^\omega$, $f^{-1}(A)$ has the $\omega$-Baire property in $X$.

**Remark**
1. Every universally Baire set is Lebesgue measurable and it has the Baire property in $2^\omega$.
2. The collection of universally Baire sets forms a $\sigma$-algebra containing all the open sets in $2^\omega$. So every Borel set is universally Baire.
Universally Baire sets; The case for subsets of $2^\omega$

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Let $A \subseteq 2^\omega$. We say $A$ is **universally Baire** if for any compact Hausdorff $X$ and any continuous $f : X \rightarrow 2^\omega$, $f^{-1}(A)$ has the $\omega$-Baire property in $X$.

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**Remark**

Let $A \subseteq 2^\omega$. The following are equivalent:

- $A$ is universally Baire, and
- for every $B$ and every continuous $f : St(B) \rightarrow 2^\omega$, $f^{-1}(A)$ has the $\omega$-Baire property in $St(B)$.
Universally Baire sets; The case for subsets of $2^\kappa$

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Universally Baire sets; The case for subsets of $2^\kappa$

### Two parts of generalization

From $\omega$ to $\kappa$, and from ZFC to $T \supseteq ZFC$.

### Why from ZFC to $T \supseteq ZFC$?

Let $\kappa = \omega_1$. Then ZFC is not strong enough to decide the theory of $(\omega_1, P(\omega_1), \in)$.

### Definition

Let $T \supseteq ZFC$. A subset $A$ of $2^\kappa$ is **universally Baire in $2^\kappa$ in $T$** ($uB^T_\kappa$) if for all $B$ with $B \models T$, and for all continuous $f : St(B) \to 2^\kappa$, $f^{-1}(A)$ has the $\kappa$-Baire property in $St(B)$. 
Recall

A subset $A$ of $2^\kappa$ is $uB^T_\kappa$ if for all $B$ with $\models_B T$ and for all continuous $f : St(B) \to 2^\kappa$, $f^{-1}(A)$ has the $\kappa$-Baire property in $St(B)$.

Example

We work in $T$.

1. Let $T = ZFC + V = L$. Then every subset of $2^\kappa$ is $uB^T_\kappa$. 
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2. Let $\kappa = \omega_1$ and $T = \text{ZFC}$. If MM holds, then every well-order on $2^{\omega_1}$ is NOT $uB^T_{\omega_1}$. 
Universally Baire sets; Examples

Recall

A subset $A$ of $2^\kappa$ is $uB^T_\kappa$ if for all $B$ with $\models_B T$ and for all continuous $f : St(B) \to 2^\kappa$, $f^{-1}(A)$ has the $\kappa$-Baire property in $St(B)$.

Example

We work in $T$.

1. Let $T = ZFC + V = L$. Then every subset of $2^\kappa$ is $uB^T_\kappa$.
2. Let $\kappa = \omega_1$ and $T = ZFC$. If MM holds, then every well-order on $2^{\omega_1}$ is NOT $uB^T_{\omega_1}$.
3. Let $\kappa = \omega_1$ and $T = ZFC + MM$. Working in $T$, there is a well-order on $2^{\omega_1}$ definable over the structure $(\omega_1, P(\omega_1), \epsilon)$ which is $uB^T_{\omega_1}$. 
Universally Baire sets; Result 1

**Theorem**

Let $\kappa = \omega_1$ and $T = \text{ZFC} + \text{Large Cardinals} + \text{MM}^{++}$. Then working in $T$,

1. every subset of $2^{\omega_1}$ definable in the structure $(\omega_1, \mathcal{P}(\omega_1), \in)$ is $\text{uB}^T_{\omega_1}$, and
2. moreover, every subset of $2^{\omega_1}$ which is in $L(\mathcal{P}(\omega_1))$ is $\text{uB}^T_{\omega_1}$. 

The above theorem is analogous to the following:

**Theorem (Woodin)**

Assuming large cardinals,

1. every subset of $2^{\omega_1}$ definable in the 2nd order structure $(\omega_1, \mathcal{P}(\omega_1), \in)$ is universally Baire in $2^{\omega_1}$, and
2. moreover, every subset of $2^{\omega_1}$ which is in $L(\mathcal{P}(\omega_1))$ is universally Baire in $2^{\omega_1}$. 

Universally Baire sets; Result 1

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Let $\kappa = \omega_1$ and $T = \text{ZFC} + \text{Large Cardinals} + \text{MM}^{+++}$. Then working in $T$,

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2. moreover, every subset of $2^\omega$ which is in $L(\mathcal{P}(\omega_1))$ is universally Baire in $2^\omega$. 
Universally Baire sets; Tree representation

Theorem (Feng, Magidor, and Woodin)

Let $A \subseteq 2^\omega$. Then the following are equivalent:

1. $A$ is universally Baire in $2^\omega$, and
2. for all $B$, there are a set $Y$ and trees $S_1, S_2$ on $2 \times Y$ such that $A = p[S_1]$ and $\models_B \text{“} p[\check{S}_1] = 2^\omega \setminus p[\check{S}_2] \text{”}.$

Notation

For a set $Y$, let $Y^{<\omega} = \bigcup_{n \in \omega} Y^n$.

Definition

1. Let $Y$ be a set. A subset $S$ of $Y^{<\omega}$ is a tree if $S$ is closed under initial segments, i.e., if $s$ is in $S$ and $t \subseteq s$, then $t$ is also in $S$.
2. For a tree $S$ on $Y$, $[S] = \{ x \in Y^\omega \mid (\forall n \in \omega) \ x \upharpoonright n \in S \}$.
3. For a tree $S$ on $2 \times Y$, $p[S] = \{ x \in 2^\omega \mid (\exists y \in Y^\omega) \ (x, y) \in [S] \}.$
Theorem
Let $T \supseteq ZFC$ and $A \subseteq 2^\kappa$. Then the following are equivalent:

1. $A$ is $uB_T^\kappa$, and

2. for all $B$ with $FA_\kappa(B)$ and $\models_B T$, there are a set $Y$ and $\text{tree}^\kappa S_1, S_2$ on $2 \times Y$ such that $A = p[S_1]$ and $\models_B \text{"} p[\bar{S}_1] = 2^\kappa \setminus p[\bar{S}_2]\text{"}$.

Notation
Let $[\kappa]^{<\omega}$ be the collection of finite subsets of $\kappa$.
For a set $Y$, let $\text{Fn}(\kappa, Y) = \{s \mid s: \text{dom}(s) \rightarrow Y \text{ and dom}(s) \in [\kappa]^{<\omega}\}$.

Definition
1. Let $Y$ be a set. A subset $S$ of $\text{Fn}(\kappa, Y)$ is a $\text{tree}^\kappa$ if $S$ is closed under initial segments, i.e., if $s$ is in $S$ and $t \subseteq s$, then $t$ is also in $S$.
2. For a $\text{tree}^\kappa S$ on $Y$, $[S] = \{x \in Y^\kappa \mid (\forall u \in [\kappa]^{<\omega}) x \upharpoonright u \in S\}$. 
Let $T = \text{ZFC}$. Assuming large cardinals and the \textit{generic nice UBH}, for any $Z \prec_{\Sigma_{2014}} V$ of size $\omega_1$, if $M$ is the transitive collapse of $Z$, then $M$ is iterable via an iteration strategy coded by a $\text{uB}_{\omega_1}^T$ set.
Universally Baire sets; Result 2

**Theorem**

Let $T = \text{ZFC}$. Assuming large cardinals and the generic nice UBH, for any $Z \prec_{\Sigma_{2014}} V$ of size $\omega_1$, if $M$ is the transitive collapse of $Z$, then $M$ is iterable via an iteration strategy coded by a $uB^T_{\omega_1}$ set.

The above Theorem is analogous to the following:

**Theorem (Woodin)**

Assuming large cardinals and the generic nice UBH, for any $Z \prec_{\Sigma_{2014}} V$ of size $\omega$, if $M$ is the transitive collapse of $Z$, then $M$ is iterable via an iteration strategy coded by a universally Baire set in $2^\omega$. 
Universally Baire sets; Beyond generic absoluteness

Using iterable structures of size \( \omega \), one can prove the following:

**Theorem (Woodin)**

Assuming large cardinals and the generic nice UBH, if a \( \Delta^\text{ZFC}_2 \) formula is generically absolute, then it is honestly absolute.

**Definition**

Let \( \phi \) be a formula and \( x \) be a subset of \( \omega \).

1. \( \phi[x] \) is generically absolute if \( V \models \phi[x] \iff V^B \models \phi[x] \) for any \( B \),
2. \( \phi[x] \) is honestly absolute if \( V \models \phi[x] \iff W \models \phi[x] \) for any \( \omega \)-model \( W \) of ZFC with the following condition:

For any universally Baire set \( A \) in \( 2^\omega \) in \( V \), there is a universally Baire set \( A' \) in \( W \) such that \( (\omega, P(\omega), \in, A)^V \prec_2 (\omega, P(\omega), \in, A')^W \).
Conjecture

Let $\kappa = \omega_1$ and $T = \text{ZFC} + \text{Large Cardinals} + \text{MM}^{+++}$. We work in $T$ and assume that the generic nice UBH holds. Then if a $\Delta^T_2$ formula is generically absolute in $T$, then it is honestly absolute in $T$.

Definition

Let $\phi$ be a formula and $x$ be a subset of $\omega_1$.

1. $\phi[x]$ is generically absolute in $T$ if $V \models \phi[x] \iff V^B \models \phi[x]$ for any $B$ with $\text{FA}_{\omega_1}(B)$ and $\models_B T$.

2. $\phi[x]$ is honestly absolute in $T$ if $V \models \phi[x] \iff W \models \phi[x]$ for any model $W$ of $T$ with the following conditions:
   - $(\omega_1, \in)^V = (\omega_1, \in)^W$, and
   - for any $uB^T_{\omega_1}$ set $A$ in $V$, there is a $uB^T_{\omega_1}$ set $A'$ in $W$ such that $(\omega_1, \mathcal{P}(\omega_1), \in, A)^V \prec_2 (\omega_1, \mathcal{P}(\omega_1), \in, A')^W$. 