The theory of universally Baire sets in 2^{ω_1}

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From now on...

We work in ZFC unless clearly specified.

Goal & Today's topic

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Understand the theory of subsets of ω_1 under "ZFC + Large Cardinals + Forcing Axioms" as much as the theory of subsets of ω ("reals") under "ZFC + Large Cardinals".

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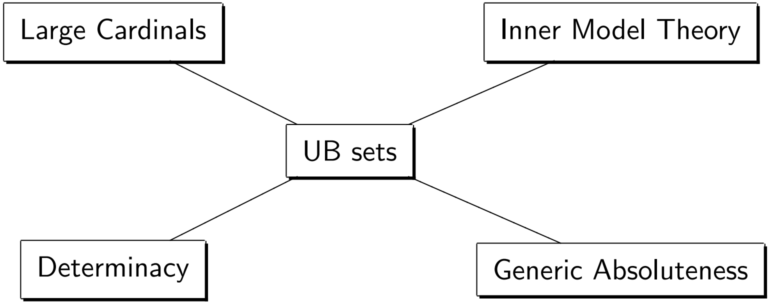
Today

Will generalize the notion of universally Baireness for subsets of $\mathcal{P}(\omega)$ ("sets of reals") to that for subsets of $\mathcal{P}(\omega_1)$.

Motivation; Universally Baire sets of reals



Background; Continuum Hypothesis and Gödel

Theorem (Cantor)

The size of the continuum is bigger than $\aleph_0$.

Definition

The Continuum Hypothesis states that the size of the continuum is $\aleph_1$.

Theorem (Gödel)

In ZFC, $\mathrm{L} \models$ "ZFC + CH". In particular, one cannot refute CH in ZFC.

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In ZFC, $\mathrm{L} \models$ "ZFC + CH". In particular, one cannot refute CH in ZFC.

Gödel's expectation before forcing

Use large cardinals to decide the truth-value of CH.

Background; Continuum Hypothesis and Large Cardinals

Gödel's Program

Decide the truth-values of mathematically interesting statements independent of ZFC in "well-justified" extensions of ZFC.

Background; Continuum Hypothesis and Large Cardinals

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Decide the truth-values of mathematically interesting statements independent of ZFC in "well-justified" extensions of ZFC.

Theorem (Cohen)

Using forcing, one can show that one cannot prove CH in ZFC.

Theorem (Levy, Solovay)

Using forcing, one can show that large cardinals cannot decide the truth-value of CH in ZFC.

Background; Gödel's Program and Large Cardinals

Theorem (Shelah, Woodin)

Assuming large cardinals, every set of reals in $\mathrm{L}(\mathbb{R})$ is Lebesgue measurable.

Theorem (Woodin)

Assuming large cardinals, the 1st-order theory of $(\mathrm{L}(\mathbb{R}), \in)$ is invariant under set generic extensions, i.e., for all 1st-order sentences ϕ, $(\mathrm{L}(\mathbb{R}), \in)^V \vDash \phi \iff (\mathrm{L}(\mathbb{R}), \in)^{V^P} \vDash \phi$ for all partial orders P.

Background; Forcing Axioms

Forcing Axioms are a generalization of Baire Category Theorem:

Baire Category Theorem

If X is either a compact Hausdorff space or a completely metrizable space, then the intersection of countably many dense open sets is non-empty.

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Baire Category Theorem

If X is either a compact Hausdorff space or a completely metrizable space, then the intersection of countably many dense open sets is non-empty.

Convention

From now on,

- κ will always be an infinite cardinal, and
- X will always be a topological space.

Definition

Let $\mathrm{BC}_\kappa(X)$ state that the intersection of κ-many dense open sets in X is non-empty.

Background; Forcing Axioms ctd.

Convention

From now on, B will always be a complete Boolean algebra.

Definition

The Stone space of B (denoted by $\mathrm{St}(B)$) is the collection of ultrafilters on B topologized by the sets $O_b = \{G \in \mathrm{St}(B) \mid b \in G\}$ for $b \in B$.

Remark

Every Stone space $\mathrm{St}(B)$ is a compact Hausdorff space.

Background; Forcing Axioms ctd.

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Every Stone space $\mathrm{St}(B)$ is a compact Hausdorff space.

Definition

The forcing axiom for B at κ (denoted by $\mathrm{FA}_\kappa(B)$) states that $\mathrm{BC}_\kappa(X)$ holds for $X = \mathrm{St}(B)$.

Background; Forcing Axioms ctd..

Remark

1. For any compact Hausdorff X, there exists a B such that
$$\mathrm{BC}_\kappa(X) \iff \mathrm{FA}_\kappa(B).$$
2. Let $\kappa = \omega_1$. There is a B such that $\mathrm{FA}_{\omega_1}(B)$ fails. In fact, if $\mathrm{FA}_{\omega_1}(B)$ holds, then B must be stationary set preserving.

Background; Forcing Axioms ctd..

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Definition

The axiom Martin's Maximum (MM) states that $\mathrm{FA}_{\omega_1}(B)$ holds for all B which are stationary set preserving.

Note: The axiom MM^{+++} is a technical strengthening of MM.

Background; Forcing Axioms ctd...

Theorem (Foreman, Magidor, and Shelah)

1. The theory ZFC + MM is consistent assuming the consistency of ZFC + Large Cardinals.
2. MM implies that the size of the continuum is $\aleph_2$.

Background; Forcing Axioms ctd...

Theorem (Foreman, Magidor, and Shelah)

1. The theory ZFC + MM is consistent assuming the consistency of ZFC + Large Cardinals.
2. MM implies that the size of the continuum is $\aleph_2$.

Theorem (Viale)

Assuming large cardinals and MM^{+++}, the following holds: For any B which is stationary set preserving with $\Vdash_B \mathrm{MM}^{+++}$,

$$(\omega_1, \mathcal{P}(\omega_1), \in)^V \prec_{\mathbf{\Sigma}_\omega} (\omega_1, \mathcal{P}(\omega_1), \in)^{V^B}.$$

Furthermore,

$$(\mathrm{L}(\mathrm{Ord}^{\omega_1}), \in)^V \equiv (\mathrm{L}(\mathrm{Ord}^{\omega_1}), \in)^{V^B}.$$

Universally Baire sets; Preparation

Convention

1. We will identify $\mathcal{P}(\kappa)$ with $2^\kappa = \{x \mid x\colon \kappa \to 2\}$.
2. We will consider 2^κ as the product space of the discrete space 2.

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Why consider the product space on 2^κ?

Want to generalize the correspondence $(f \mapsto \tau_f, \tau \mapsto f_\tau)$ between continuous functions from $St(B)$ to 2^ω and B-names for a subset of ω with the following properties:

1. $f_{(\tau_f)} = f$ and $\Vdash_B \tau_{(f_\tau)} = \tau$, and
2. for any $Z \prec_{\Sigma_{2014}} V$ with $B, f, \tau \in Z$, and any (Z, B)-generic g, $\mathrm{val}_g(\tau_f) = f(g)$ and $f_\tau(g) = \mathrm{val}_g(\tau)$.

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Observation

Using the product topology on 2^κ, one can generalize the above correspondence to the one in the context of 2^κ.

Universally Baire sets; Preparation ctd.

Definition

Let $A \subseteq X$.

1. The set A is nowhere dense if A is disjoint from an open dense subset of X.
2. The set A is κ-meager if it is the union of κ-many nowhere dense sets in X.

Universally Baire sets; Preparation ctd.

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Baire Category Theorem, reformulated

Let $\kappa = \omega$. If X is either compact Hausdorff or completely metrizable, then X is NOT ω-meager.

Remark

The space X is κ-meager if and only if $\mathrm{BC}_\kappa(X)$ *fails*.

Universally Baire sets; Preparation ctd..

Definition

Let $A \subseteq X$. The set A has the κ-Baire property in X if there is an open set U in X such that $U \triangle A = (U \setminus A) \cup (A \setminus U)$ is κ-meager.

Remark

1. The collection of subsets of X with the κ-Baire property is closed under complements and unions of κ-many sets, and it contains all the open sets in X.
2. If X is κ-meager, then *every* subset of X has the κ-Baire property.

Universally Baire sets; The case for subsets of 2^ω

Definition (Feng, Magidor, and Woodin)

Let $A \subseteq 2^\omega$. We say A is universally Baire if for any compact Hausdorff X and any continuous $f: X \to 2^\omega$, $f^{-1}(A)$ has the ω-Baire property in X.

Remark

1. Every universally Baire set is Lebesgue measurable and it has the Baire property in 2^ω.
2. The collection of universally Baire sets forms a σ-algebra containing all the open sets in 2^ω. So every Borel set is universally Baire.

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Remark

Let $A \subseteq 2^\omega$. The following are equivalent:

- A is universally Baire, and
- for every B and every continuous $f\colon St(B) \to 2^\omega$, $f^{-1}(A)$ has the ω-Baire property in $St(B)$.

Universally Baire sets; The case for subsets of 2^κ

Two parts of generalization

From ω to κ, and from ZFC to T $\supseteq$ ZFC.

Why from ZFC to T $\supseteq$ ZFC?

Let $\kappa = \omega_1$. Then ZFC is not strong enough to decide the theory of $(\omega_1, \mathcal{P}(\omega_1), \in)$.

Universally Baire sets; The case for subsets of 2^κ

Two parts of generalization

From ω to κ, and from ZFC to $\mathrm{T} \supseteq \mathrm{ZFC}$.

Why from ZFC to $\mathrm{T} \supseteq \mathrm{ZFC}$?

Let $\kappa = \omega_1$. Then ZFC is not strong enough to decide the theory of $(\omega_1, \mathcal{P}(\omega_1), \in)$.

Definition

Let $\mathrm{T} \supseteq \mathrm{ZFC}$. A subset A of 2^κ is universally Baire in 2^κ in T ($\mathrm{uB}^{\mathrm{T}}_\kappa$) if for all B with $\Vdash_B \mathrm{T}$, and for all continuous $f : St(B) \to 2^\kappa$, $f^{-1}(A)$ has the κ-Baire property in $St(B)$.

Universally Baire sets; Examples

Recall

A subset A of 2^κ is $\mathrm{uB}_\kappa^{\mathsf{T}}$ if for all B with $\Vdash_B \mathsf{T}$ and for all continuous $f\colon St(B) \to 2^\kappa$, $f^{-1}(A)$ has the κ-Baire property in $St(B)$.

Example

We work in T.

1. Let $\mathsf{T} = \mathsf{ZFC} + V = \mathrm{L}$. Then *every* subset of 2^κ is $\mathrm{uB}_\kappa^{\mathsf{T}}$.

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Recall

A subset A of 2^κ is $\mathrm{uB}^{\mathsf{T}}_\kappa$ if for all B with $\Vdash_B \mathsf{T}$ and for all continuous $f\colon St(B) \to 2^\kappa$, $f^{-1}(A)$ has the κ-Baire property in $St(B)$.

Example

We work in T.

1. Let $\mathsf{T} = \mathsf{ZFC} + V = \mathrm{L}$. Then *every* subset of 2^κ is $\mathrm{uB}^{\mathsf{T}}_\kappa$.
2. Let $\kappa = \omega_1$ and $\mathsf{T} = \mathsf{ZFC}$. If MM holds, then every well-order on 2^{ω_1} is NOT $\mathrm{uB}^{T}_{\omega_1}$.

Universally Baire sets; Examples

Recall

A subset A of 2^κ is $\mathrm{uB}_\kappa^{\mathsf{T}}$ if for all B with $\Vdash_B \mathsf{T}$ and for all continuous $f: St(B) \to 2^\kappa$, $f^{-1}(A)$ has the κ-Baire property in $St(B)$.

Example

We work in T.

1. Let $\mathsf{T} = \mathsf{ZFC} + V = \mathrm{L}$. Then *every* subset of 2^κ is $\mathrm{uB}_\kappa^{\mathsf{T}}$.
2. Let $\kappa = \omega_1$ and $\mathsf{T} = \mathsf{ZFC}$. If MM holds, then every well-order on 2^{ω_1} is NOT $\mathrm{uB}_{\omega_1}^{T}$.
3. Let $\kappa = \omega_1$ and $\mathsf{T} = \mathsf{ZFC} + \mathsf{MM}$. Working in T, there is a well-order on 2^{ω_1} definable over the structure $(\omega_1, \mathcal{P}(\omega_1), \in)$ which is $\mathrm{uB}_{\omega_1}^{T}$.

Universally Baire sets; Result 1

Theorem

Let $\kappa = \omega_1$ and $\mathsf{T} = \mathsf{ZFC} + \text{Large Cardinals} + \mathsf{MM}^{+++}$.
Then working in T,

1. every subset of 2^{ω_1} definable in the structure $(\omega_1, \mathcal{P}(\omega_1), \in)$ is $\mathsf{uB}^T_{\omega_1}$, and
2. moreover, every subset of 2^{ω_1} which is in $\mathrm{L}(\mathcal{P}(\omega_1))$ is $\mathsf{uB}^T_{\omega_1}$.

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Let $\kappa = \omega_1$ and $\mathsf{T} = \mathsf{ZFC} + \text{Large Cardinals} + \mathsf{MM}^{+++}$.
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2. moreover, every subset of 2^{ω_1} which is in $\mathrm{L}(\mathcal{P}(\omega_1))$ is $\mathsf{uB}^T_{\omega_1}$.

The above theorem is analogous to the following:

Theorem (Woodin)

Assuming large cardinals,

1. every subset of 2^ω definable in the 2nd order structure $(\omega, \mathcal{P}(\omega), \in)$ is universally Baire in 2^ω, and
2. moreover, every subset of 2^ω which is in $\mathrm{L}(\mathbb{R})$ is universally Baire in 2^ω.

Universally Baire sets; Tree representation

Theorem (Feng, Magidor, and Woodin)

Let $A \subseteq 2^\omega$. Then the following are equivalent:

1. A is universally Baire in 2^ω, and
2. for all B, there are a set Y and trees S_1, S_2 on $2 \times Y$ such that $A = \mathrm{p}[S_1]$ and $\Vdash_B$ "$\mathrm{p}[\check{S}_1] = 2^\omega \setminus \mathrm{p}[\check{S}_2]$".

Notation

For a set Y, let $Y^{<\omega} = \bigcup_{n\in\omega} Y^n$.

Definition

1. Let Y be a set. A subset S of $Y^{<\omega}$ is a tree if S is closed under initial segments, i.e., if s is in S and $t \subseteq s$, then t is also in S.
2. For a tree S on Y, $[S] = \{x \in Y^\omega \mid (\forall n \in \omega)\ x \restriction n \in S\}$.
3. For a tree S on $2 \times Y$, $\mathrm{p}[S] = \{x \in 2^\omega \mid (\exists y \in Y^\omega)\ (x, y) \in [S]\}$.

Universally Baire sets; Tree representation ctd.

Theorem

Let T $\supseteq ZFC$ and $A \subseteq 2^\kappa$. Then the following are equivalent:

1. A is uB_κ^T, and
2. for all B with $\mathrm{FA}_\kappa(B)$ and $\Vdash_B$ T, there are a set Y and tree$^\kappa$ S_1, S_2 on $2 \times Y$ such that $A = \mathrm{p}[S_1]$ and $\Vdash_B$ "$\mathrm{p}[\check{S}_1] = 2^\kappa \setminus \mathrm{p}[\check{S}_2]$".

Notation

Let $[\kappa]^{<\omega}$ be the collection of finite subsets of κ.
For a set Y, let $\mathrm{Fn}(\kappa, Y) = \{s \mid s\colon \mathrm{dom}(s) \to Y \text{ and } \mathrm{dom}(s) \in [\kappa]^{<\omega}\}$.

Definition

1. Let Y be a set. A subset S of $\mathrm{Fn}(\kappa, Y)$ is a tree$^\kappa$ if S is closed under initial segments, i.e., if s is in S and $t \subseteq s$, then t is also in S.
2. For a tree$^\kappa$ S on Y, $[S] = \{x \in Y^\kappa \mid (\forall u \in [\kappa]^{<\omega})\ x \restriction u \in S\}$.

Universally Baire sets; Result 2

Theorem

Let $\mathrm{T} = \mathrm{ZFC}$. Assuming large cardinals and the generic nice UBH, for any $Z \prec_{\mathbf{\Sigma}_{2014}} V$ of size ω_1, if M is the transitive collapse of Z, then M is iterable via an iteration strategy coded by a $\mathrm{uB}^{\mathrm{T}}_{\omega_1}$ set.

Universally Baire sets; Result 2

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The above Theorem is analogous to the following:

Theorem (Woodin)

Assuming large cardinals and the generic nice UBH, for any $Z \prec_{\boldsymbol{\Sigma}_{2014}} V$ of size ω, if M is the transitive collapse of Z, then M is iterable via an iteration strategy coded by a universally Baire set in 2^ω.

Universally Baire sets; Beyond generic absoluteness

Using iterable structures of size ω, one can prove the following:

Theorem (Woodin)

Assuming large cardinals and the generic nice UBH, if a Δ_2^{ZFC} formula is generically absolute, then it is honestly absolute.

Definition

Let ϕ be a formula and x be a subset of ω.

1. $\phi[x]$ is generically absolute if $V \models \phi[x] \iff V^B \models \phi[x]$ for any B,
2. $\phi[x]$ is honestly absolute if $V \models \phi[x] \iff W \models \phi[x]$ for any ω-model W of ZFC with the following condition:

   For any universally Baire set A in 2^ω in V, there is a universally Baire set A' in W such that $(\omega, \mathcal{P}(\omega), \in, A)^V \prec_{\boldsymbol{\Sigma}_2} (\omega, \mathcal{P}(\omega), \in, A')^W$.

Universally Baire sets; Beyond generic absoluteness ctd.

Conjecture

Let $\kappa = \omega_1$ and $\mathsf{T} = \mathsf{ZFC} + \text{Large Cardinals} + \mathsf{MM}^{+++}$. We work in T and assume that the generic nice UBH holds. Then if a Δ_2^{T} formula is generically absolute in T, then it is honestly absolute in T.

Definition

Let ϕ be a formula and x be a subset of ω_1.

1. $\phi[x]$ is generically absolute in T if $V \vDash \phi[x] \iff V^B \vDash \phi[x]$ for any B with $\mathsf{FA}_{\omega_1}(B)$ and $\Vdash_B \mathsf{T}$.
2. $\phi[x]$ is honestly absolute in T if $V \vDash \phi[x] \iff W \vDash \phi[x]$ for any model W of T with the following conditions:
   - $(\omega_1, \in)^V = (\omega_1, \in)^W$, and
   - for any $\mathsf{uB}_{\omega_1}^{\mathsf{T}}$ set A in V, there is a $\mathsf{uB}_{\omega_1}^{\mathsf{T}}$ set A' in W such that $(\omega_1, \mathcal{P}(\omega_1), \in, A)^V \prec_{\boldsymbol{\Sigma}_2} (\omega_1, \mathcal{P}(\omega_1), \in, A')^W$.