Intuitionistic provability and uniformly provability in RCA

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1. Introduction.
2. Motivating Results.
3. Investigations.
Constructivity

The notion of constructivity has been interested in foundation of mathematics.
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- The first problem might be how to formulate constructivity in mathematics.
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- The first problem might be how to formulate constructivity in mathematics.
- In this talk, we think of constructivity as (Turing) computational algorithm and show some equivalences between global and local constructive provability with respect to reverse mathematics.
Global Constructivity: Constructive Mathematics

Constructive mathematics was initiated mainly by L.E.J. Brouwer based on his philosophy in the disputation on foundation of mathematics in the early 20th century.
The following is an exposition from


Constructive mathematics is distinguished from its traditional counterpart, classical mathematics, by the strict interpretation of the phrase “there exists” as “we can construct”. In order to work constructively, we need to re-interpret not only the existential quantifier but all the logical connectives and quantifiers as instructions on how to construct a proof of the statement involving these logical expressions.
Intuitionistic Two-sorted Arithmetic

Intuitionistic two-sorted arithmetic EL introduced by A. S. Troelstra in 1970’s is served as base theory formalizing constructive mathematics.
Intuitionistic Two-sorted Arithmetic

- As language, EL has two-sorted variables (for numbers and functions), 0, successor \( S \), abstraction operators \( \lambda x. \) (only for numbers), a recursor \( R \), function constants for all primitive recursive functions and equality \( = \) for numbers.
- Terms of EL are defined in the usual manner.
- Axioms and rules of EL include
  - \( \lambda \)-CON: \( (\lambda x.t) \, t' = t[t'/x] \)
  - REC: \( R \varphi 0 = 0 \) and \( R \varphi (S \, t') = \varphi (R \varphi t', t') \)
  - IND: \( A(0) \land \forall x (A(x) \rightarrow A(Sx)) \rightarrow \forall x A(x) \)
  - QF-AC\(^{0,0} \): \( \forall x \exists y A_{qf}(x, y) \rightarrow \exists f \forall x A_{qf}(x, fx) \)
- EL does not have the law-of-excluded-middle: \( A \lor \neg A \).
Intuitionistic Two-sorted Arithmetic

- As language, EL has two-sorted variables (for numbers and functions), 0, successor $S$, abstraction operators $\lambda x.$ (only for numbers), a recuuror $R$, function constants for all primitive recursive functions and equality $=$ for numbers.
- Terms of EL are defined in the usual manner.
- Axioms and rules of EL include
  - $\lambda$-CON: $(\lambda x. t)t' = t[t'/x]$
  - REC: $Rt\varphi 0 = 0$ and $Rt\varphi (S t') = \varphi (Rt\varphi t', t')$
  - IND: $A(0) \land \forall x (A(x) \rightarrow A(Sx)) \rightarrow \forall x A(x)$
  - QF-AC$^{0,0}$: $\forall x \exists y A_{qf} (x, y) \rightarrow \exists f \forall x A_{qf} (x,fx)$
- EL does not have the law-of-excluded-middle: $A \lor \neg A$.

Remark.

$\text{EL} \vdash A \lor B \leftrightarrow \exists k (k = 0 \rightarrow A \land k \neq 0 \rightarrow B)$. 
## Intuitionistic & Classical Systems

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<td>HA</td>
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<td>RCA ( = EL + LEM)</td>
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<td>EL$_0$</td>
<td>RCA$_0$ ( = EL$_0$ + LEM)</td>
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- One can identify EL + LEM with function-based language as RCA (RCA$_0$ + full induction) with set-based language, since $\Delta^0_1$-CA (by function-based language) is derived from QF-AC$^{0,0}$ and LEM.

- One can identify EL$_0$(with QF-IND)+LEM as RCA$_0$, since $\Sigma^0_1$-IND is intuitionistically derived from QF-AC$^{0,0}$ and QF-IND intuitionistically.
Intuitionistic & Classical Systems

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- One can identify EL + LEM with function-based language as RCA (RCA₀ + full induction) with set-based language, since $\Delta_1^0$-CA (by function-based language) is derived from $QF$-$AC^{0,0}$ and LEM.
- One can identify EL₀(with QF-IND)+LEM as RCA₀, since $\Sigma_1^0$-IND is intuitionistically derived from $QF$-$AC^{0,0}$ and QF-IND intuitionistically.

Then we shall use the same notations RCA and RCA₀ respectively for EL + LEM and EL₀ + LEM.
Local Constructivity for Mathematical Statements

Many mathematical statements have \( \Pi_2 \) form:

\[
\forall X \ (A(X) \rightarrow \exists Y B(X, Y)).
\]

**Intermediate Value Theorem.**

For any continuous function \( f : [0, 1] \rightarrow \mathbb{R} \) s.t. \( f(0) < 0 < f(1) \), then there exists a point \( m \in [0, 1] \) s.t. \( f(m) = 0 \).
Many mathematical statements have $\Pi_2$ form:

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Many $\Pi^1_2$ statements are provable in RCA (even in RCA$_0$).
Sequential Version

- Many $\Pi^1_2$ statements are provable in RCA (even in RCA$_0$).
- In some of their proofs, however, the construction of the solution $Y$ from given $X$ is not uniform.
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- To reveal the non-uniformity, the following sequential version has been investigated.

$$\forall \langle X_n \rangle_{n \in \mathbb{N}} (\forall n A(X_n) \rightarrow \exists \langle Y_n \rangle_{n \in \mathbb{N}} \forall n B(X_n, Y_n)).$$
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Examples.

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<tr>
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<tr>
<td>JD (The existence of Jordan decomposition for real square matrices)</td>
<td>RCA</td>
<td>ACA</td>
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<td>RT$^1$ (Infinite pigeonhole principle)</td>
<td>RCA</td>
<td>ACA</td>
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<td>IVT (Intermediate value theorem)</td>
<td>RCA</td>
<td>WKL</td>
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In this 10 years, it had been found that the law-of-excluded-middle (LEM) has an arithmetical hierarchy (over intuitionistic system like EL) as in reverse mathematics.

\[ \sum^0_1 \text{-LEM} \]
\[ \Pi^0_1 \text{-LEM} \]
\[ M^0 \]
\[ \Sigma^0_1 \text{-DML} \]
\[ \Pi^0_1 \text{-DML} \]

- \( M^0 \): \( \neg \neg \exists x A_{qf} \rightarrow \exists x A_{qf} \)
- \( \Sigma^0_1 \text{-LEM} \): \( \exists x A_{qf} \lor \neg \exists x A_{qf} \)
- \( \Sigma^0_1 \text{-DML} \): \( \neg (\exists x A_{qf} \land \exists y B_{qf}) \rightarrow (\neg \exists x A_{qf} \lor \neg \exists y B_{qf}) \)
In this 10 years, it had been found that the law-of-excluded-middle (LEM) has an arithmetical hierarchy (over intuitionistic system like EL) as in reverse mathematics.

\[
\Sigma^0_1\text{-LEM} \quad \Pi^0_1\text{-LEM} \\
M^0 \quad \Sigma^0_1\text{-DML} \quad \Pi^0_1\text{-DML}
\]

- \( M^0 : \neg \neg \exists x A_{qf} \rightarrow \exists x A_{qf} \)
- \( \Sigma^0_1\text{-LEM} : \exists x A_{qf} \lor \neg \exists x A_{qf} \)
- \( \Sigma^0_1\text{-DML} : \neg (\exists x A_{qf} \land \exists y B_{qf}) \rightarrow (\neg \exists x A_{qf} \lor \neg \exists y B_{qf}) \)

Recently, constructive reverse mathematic, which classify mathematical principles by that hierarchy, has been carried out.
Constructive and Sequential Reverse Mathematics

There are some corresponding results between constructive and sequential reverse mathematics.
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- **TRIC**: \( \forall \alpha \in \mathbb{R} \ (\alpha < 0 \lor \alpha = 0 \lor \alpha > 0) \).
- **DIC**: \( \forall \alpha \in \mathbb{R} \ (\alpha \leq 0 \lor \alpha \geq 0) \).

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<td>Seq(DIC) ( \leftrightarrow ) WKL.</td>
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Constructive and Sequential Reverse Mathematics

There are some corresponding results between constructive and sequential reverse mathematics.

- **TRIC**: $\forall \alpha \in \mathbb{R} (\alpha < 0 \lor \alpha = 0 \lor \alpha > 0)$.
- **DIC**: $\forall \alpha \in \mathbb{R} (\alpha \leq 0 \lor \alpha \geq 0)$.

### Fact.

**Over EL,**

- TRIC $\leftrightarrow \Sigma^0_1$-LEM.
- DIC $\leftrightarrow \Sigma^0_1$-DML.

### Fact.

**Over RCA,**

- Seq(TRIC) $\leftrightarrow$ ACA.
- Seq(DIC) $\leftrightarrow$ WKL.

### Proposition. (Ishihara 2005)

- EL $\vdash$ ACA $\leftrightarrow \Sigma^0_1$-LEM $+ \Pi^0_1$-AC$^{0,0}$.
- EL $\vdash$ WKL $\leftrightarrow \Sigma^0_1$-DML $+ \Pi^0_1$-AC$^\vee$. 
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Let us consider a \( \Pi^1_2 \) statement

\[
\forall X^1 (A(X) \rightarrow \exists Y^1 B(X, Y)).
\]

- Its provability in RCA corresponds to \( Y \) is Muchnik reducible to \( X \), i.e. for all \( X \) satisfying \( A(X) \), there is a program \( \Phi \) of Turing machine with oracle s.t. \( \Phi^X \) compute \( Y \) satisfying \( B(X, Y) \).
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- On the other hand, what one intend to represent by its sequential provability in RCA is that $Y$ is Medvedev reducible to $X$, i.e. there is a uniform program $\Phi$ of Turing machine with oracle s.t. for all $X$ satisfying $A(X)$, $\Phi^X$ compute $Y$ satisfying $B(X, Y)$
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Thus, if its sequential version derives WKL or ACA, then there is no uniform program $\Phi$ of Turing machine with oracle s.t. for all $X$ satisfying $A(X)$, $\Phi^X$ compute $Y$ satisfying $B(X, Y)$
By the way, what is the formal representation to capture precisely the uniform provability in RCA?

Two candidates;

1. There exists a (primitive recursive) term $t_1$ of RCA s.t.
   \[ \text{RCA} \vdash \forall X (A(X) \Rightarrow B(X; t_X)) \]
   where $j$ is the partial continuous operation from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$.

2. There exists a (Gödel primitive recursive) term $t_1$ of RCA! s.t.
   \[ \text{RCA!} \vdash \forall X (A(X) \Rightarrow B(X; t_X)) \]
   where RCA! $(:= E-HA! + QF-AC_1 + 0 + LEM)$ is a conservative extension of RCA in all finite types.

Remark: Both of them imply sequential provability in RCA.
By the way, what is the formal representation to capture precisely the uniform provability in RCA?

Two candidates;

1. There exists a (primitive recursive) term $t^1$ of RCA s.t.

   $\text{RCA} \vdash \forall X (A(X) \to t|X \downarrow \land B(X, t|X)),$

   where $|\cdot$ is the partial continuous operation from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$. 
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2. There exists a (Gödel primitive recursive) term $t^{1\rightarrow 1}$ of $\text{RCA}^\omega$ s.t.

   $$\text{RCA}^\omega \vdash \forall X (A(X) \rightarrow B(X, tX)),$$

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   where $\text{RCA}^\omega(:= \text{E-HA}^\omega + \text{QF-AC}^{1,0} + \text{LEM})$ is a conservative extension of RCA in all finite types.

Remark: Both of them imply sequential provability in RCA.
Kleene’s Partial Continuous Operation

We use a partial operation \((\cdot)(\cdot): \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}\) to define \(|: \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}|.

For \(\alpha, \beta: \mathbb{N} \rightarrow \mathbb{N}\),

\[
\alpha(\beta) := \begin{cases} 
\alpha(\bar{\beta}n) - 1 & \text{where } n \text{ is the least } n' \text{ s.t. } \alpha(\bar{\beta}n') \neq 0. \\
\uparrow & \text{if there is no such } n'.
\end{cases}
\]

Then

\[
\alpha|\beta := \lambda n. \alpha(\langle n \rangle \circ \beta).
\]
Proposition. (Dorais 2014, via Realizability interpretation)

If \( EL + \mathcal{M}^0 \vdash \forall X^1 (A(X) \rightarrow \exists Y^1 B(X, Y)) \), then there exists a term \( t^1 \) s.t.

\[
EL \text{ (hence RCA)} \vdash \forall X (A(X) \rightarrow t|X \downarrow \land B(X, t|X)) ,
\]

provided that \( A(X) \in N_K \) and \( B(X, Y) \in L_K \).
Proposition. (Dorais 2014, via Realizability interpretation)

If $\text{EL} + M^0 \vdash \forall X^1 (A(X) \rightarrow \exists Y^1 B(X, Y))$, then there exists a term $t^1$ s.t.

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\text{EL (hence RCA)} \vdash \forall X (A(X) \rightarrow t|X \downarrow \land B(X, t|X)),
$$

provided that $A(X) \in N_K$ and $B(X, Y) \in L_K$.

- $N_K$ is the class of formulas defined inductively as;
  - $A_{qf}$, $\exists x^{\rho} A_{qf}$ are in $N_K$.
  - If $A_1, A_2$ are in $N_K$, then $A_1 \land A_2$, $A_1 \rightarrow A_2$, $\forall x^{\rho} A_1$ are in $N_K$.

- $L_K$ is the class of formulas defined inductively as;
  - $A_{qf}$ is in $L_K$.
  - If $A_1, A_2$ are in $L_K$, then $A_1 \land A_2$, $\forall x^{\rho} A_1$ and $\exists x^{\rho} A_1$ are in $L_K$.
  - If $A_1$ is in $N_K$ and $A_2$ is in $L_K$, then $A_1 \rightarrow A_2$ is in $L_K$. 
Corollary.

If $\text{EL} + M^0 \vdash \forall X^1 (A(X) \rightarrow \exists Y^1 B(X, Y))$, then

$$
\text{RCA} \vdash \forall \langle X_n \rangle_{n \in \mathbb{N}} (\forall n A(X_n) \rightarrow \exists \langle Y_n \rangle_{n \in \mathbb{N}} \forall n B(X_n, Y_n)).
$$

provided that $A(X) \in \mathbb{N}_K$ and $B(X, Y) \in L_K$. 

Remark. (Yokoyama-F. 2013) The class $\mathbb{N}_K$ cannot be extended to involve $\exists u^0 \forall v^0 A^qf$ in the previous proposition.
Corollary.

If \( EL + M^0 \vdash \forall X^1 (A(X) \rightarrow \exists Y^1 B(X, Y)) \), then

\[
RCA \vdash \forall \langle X_n \rangle_{n \in \mathbb{N}} (\forall n A(X_n) \rightarrow \exists \langle Y_n \rangle_{n \in \mathbb{N}} \forall n B(X_n, Y_n)).
\]

provided that \( A(X) \in N_K \) and \( B(X, Y) \in L_K \).

Remark. (Yokoyama-F. 2013)

The class \( N_K \) for \( A \) cannot be extended to involve \( \exists u^0 \forall v^0 A_{qf} \) in the previous proposition.
Theorem.

If there exists a term $t^1$ s.t.

$$\text{RCA} \vdash \forall X^1 (A(X) \rightarrow t|X \downarrow \land B(X, t|X)),$$

then

$$\text{EL} + M^0 \vdash \forall X (A(X) \rightarrow \exists Y B(X, Y)),$$

provided that $A(X) \in N_M$ and $B(X, Y)$ is equivalent to some formula $\forall w^0 \exists s^0 B_{qf}(X, Y, w, s)$ over $\text{EL} + M^0$.

- $N_M$ is the class of formulas defined inductively as;
  - $A_{qf}$ is in $N_M$.
  - If $A_1, A_2$ are in $N_M$, then $A_1 \land A_2, A_1 \lor A_2, \forall x^\rho A_1, \exists x^\rho A_1$ are in $N_M$.
  - If $A$ is in $N_M$, then $\forall u^0 \exists v^0 A_{qf} \rightarrow A$ is in $N_M$. 
Negative Translation

To show this theorem, we use the following negative translation.

Definition. (Kuroda 1951)

$A^N$ is defined as $A^N \equiv \neg\neg A^*$, where $A^*$ is defined by induction on the logical structure of $A$:

- $A^* \equiv A$, if $A$ is a prime formula,
- $(A \square B)^* \equiv (A^* \square B^*)$, where $\square \in \{\land, \lor, \rightarrow\}$,
- $(\exists x^\rho A)^* \equiv \exists x^\rho A^*$,
- $(\forall x^\rho A)^* \equiv \forall x^\rho \neg\neg A^*$.
Example.

- \( \Pi^0_1, \Sigma^0_0 \):
  \[
  (\forall u^0 A_{qf} \rightarrow \exists x^0 B_{qf}) \rightarrow \exists x^0 (\forall u^0 A_{qf} \rightarrow B_{qf})
  \]
  where \( A_{qf} \) does not contain \( x \) free.
Example.

- \( \text{IP}^0(\Pi_1^0, \Sigma_0^0) : \)
  \[ (\forall u^0 A_{qf} \rightarrow \exists x^0 B_{qf}) \rightarrow \exists x^0 (\forall u^0 A_{qf} \rightarrow B_{qf}) \]

where \( A_{qf} \) does not contain \( x \) free.

\[ \text{IP}(\Pi_1^0, \Sigma_0^0)^N \equiv \neg \left( (\forall u \neg \neg A_{qf} \rightarrow \exists x B_{qf}) \rightarrow \exists x (\forall u \neg \neg A_{qf} \rightarrow B_{qf}) \right), \]
Example.

- \( \text{IP}^0(\Pi^0_1, \Sigma^0_0) : \)

\[
(\forall u^0 A_{qf} \rightarrow \exists x^0 B_{qf}) \rightarrow \exists x^0 (\forall u^0 A_{qf} \rightarrow B_{qf})
\]

where \( A_{qf} \) does not contain \( x \) free.

\[
\text{IP}(\Pi^0_1, \Sigma^0_0)^N \equiv \neg \neg \left( (\forall u \neg \neg A_{qf} \rightarrow \exists x B_{qf}) \rightarrow \exists x (\forall u \neg \neg A_{qf} \rightarrow B_{qf}) \right),
\]

which is intuitionistically equivalent to

\[
(\forall u A_{qf} \rightarrow \exists x B_{qf}) \rightarrow \neg \neg \exists x (\forall u A_{qf} \rightarrow B_{qf}).
\]
Lemma.

If RCA ⊢ A, then EL + M⁰ ⊢ A^N.

Idea of Proof.

Induction on the length of the derivation. It is enough to check all the axioms and rules of RCA. Actually M⁰ is used only to derive (QF-AC⁰,⁰)^N intuitionistically from QF-AC⁰,⁰. □
Lemma.
If \( \text{RCA} \vdash A \), then \( \text{EL} + M^0 \vdash A^N \).

Idea of Proof.
Induction on the length of the derivation. It is enough to check all the axioms and rules of RCA. Actually \( M^0 \) is used only to derive \( (\text{QF-AC}^{0,0})^N \) intuitionistically from \( \text{QF-AC}^{0,0} \).

Fact.
\( \text{EL} + M^0 \vdash \text{IP}^0(\Pi_1^0, \Sigma_0^0) \).
Lemma.
If \( \text{RCA} \vdash A \), then \( \text{EL} + M^0 \vdash A \wedge N \).

Idea of Proof.
Induction on the length of the derivation. It is enough to check all the axioms and rules of RCA. Actually \( M^0 \) is used only to derive \( (\text{QF-AC}^{0,0})^N \) intuitionistically from \( \text{QF-AC}^{0,0} \).

Fact.
\( \text{EL} + M^0 \vdash \text{IP}^0(\Pi^0_1, \Sigma^0_0) \).

Proof.
\[
\text{IP}^0(\Pi^0_1, \Sigma^0_0)^N \\
\rightarrow_i (\forall uA_{qf} \rightarrow \exists xB_{qf}) \rightarrow \neg \exists x (\forall uA_{qf} \rightarrow B_{qf})
\]
Lemma.

If RCA ⊢ A, then EL + M^0 ⊢ A^N.

Idea of Proof.

Induction on the length of the derivation. It is enough to check all the axioms and rules of RCA. Actually M^0 is used only to derive (QF-AC^{0,0})^N intuitionistically from QF-AC^{0,0}.

Fact.

EL + M^0 ⊢ IP^0(Π^0_1, Σ^0_0).

Proof.

IP^0(Π^0_1, Σ^0_0)^N
→_i (∀uA_{qf} → ∃xB_{qf}) → ¬¬∃x (∀uA_{qf} → B_{qf})
→_using M^0 (∀uA_{qf} → ∃xB_{qf}) → ¬¬∃x, u (A_{qf} → B_{qf})
Lemma.

If RCA ⊩ A, then EL + M^0 ⊩ A^N.

Idea of Proof.

Induction on the length of the derivation. It is enough to check all the axioms and rules of RCA. Actually M^0 is used only to derive \((\text{QF-AC}^0,0)^N\) intuitionistically from \(\text{QF-AC}^0,0\).

Fact.

\(\text{EL + M}^0 \vdash \text{IP}^0(\Pi^0_1, \Sigma^0_0)\).

Proof.

\[
\begin{align*}
\text{IP}^0(\Pi^0_1, \Sigma^0_0)^N &\\
\rightarrow_i & (\forall uA_{qf} \rightarrow \exists x B_{qf}) \rightarrow \neg\neg\exists x (\forall uA_{qf} \rightarrow B_{qf}) \\
\rightarrow_{\text{using } M^0} & (\forall uA_{qf} \rightarrow \exists x B_{qf}) \rightarrow \neg\neg\exists x, u (A_{qf} \rightarrow B_{qf}) \\
\rightarrow_{\text{using } M^0} & (\forall uA_{qf} \rightarrow \exists x B_{qf}) \rightarrow \exists x, u (A_{qf} \rightarrow B_{qf})
\end{align*}
\]
Lemma.

If RCA \vdash A, then EL + M^0 \vdash A^N.

Idea of Proof.

Induction on the length of the derivation. It is enough to check all the axioms and rules of RCA. Actually M^0 is used only to derive \((QF-AC^{0,0})^N\) intuitionistically from QF-AC^{0,0}.

Fact.

EL + M^0 \vdash IP^0(\Pi^0_1, \Sigma^0_0).

Proof.

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\begin{align*}
\rightarrow_i & \quad IP^0(\Pi^0_1, \Sigma^0_0)^N \\
\rightarrow \text{using } M^0 & \quad (\forall uA_{qf} \rightarrow \exists xB_{qf}) \rightarrow \neg \neg \exists x (\forall uA_{qf} \rightarrow B_{qf}) \\
\rightarrow \text{using } M^0 & \quad (\forall uA_{qf} \rightarrow \exists xB_{qf}) \rightarrow \neg \neg \exists x, u (A_{qf} \rightarrow B_{qf}) \\
\rightarrow_i & \quad (\forall uA_{qf} \rightarrow \exists xB_{qf}) \rightarrow \exists x (\forall uA_{qf} \rightarrow B_{qf}).
\end{align*}
\]
Lemma.

For any formula $A \in N_M$, $EL + M_0 \vdash A \rightarrow A^*$.

Proof is by induction on the structure of $N_M$.

- $N_M$ is the class of formulas defined inductively as;
  - $A_{qf}$ is in $N_M$.
  - If $A_1, A_2$ are in $N_M$, then $A_1 \land A_2, A_1 \lor A_2, \forall x^\rho A_1, \exists x^\rho A_1$ are in $N_M$.
  - If $A$ is in $N_M$, then $\forall u^\rho \exists v^0 A_{qf} \rightarrow A$ is in $N_M$. 
Theorem.

If there exists a term $t^1$ s.t.

$$\text{RCA} \vdash \forall X^1 (A(X) \rightarrow t|X \downarrow \wedge B(X, t|X)),$$

then

$$\text{EL} + M^0 \vdash \forall X (A(X) \rightarrow \exists Y B(X, Y)),$$

provided that $A(X) \in N_M$ and $B(X, Y)$ is equivalent to some formula $\forall w^\rho \exists s^0 B_{qf}(X, Y, w, s)$ over $\text{EL} + M^0$.

Proof Sketch.
**Theorem.**

If there exists a term $t^1$ s.t.

$$\text{RCA} \vdash \forall X^1 (A(X) \rightarrow t|X \downarrow \land B(X, t|X)),$$

then

$$\text{EL} + M^0 \vdash \forall X (A(X) \rightarrow \exists YB(X, Y)),$$

provided that $A(X) \in \mathbb{N}_M$ and $B(X, Y)$ is equivalent to some formula $\forall w^\rho \exists s^0B_{qf}(X, Y, w, s)$ over $\text{EL} + M^0$.

**Proof Sketch.**

By negative translation, we have that $\text{EL} + M^0$ derives

$$\forall X^1 (A^*(X) \rightarrow \neg \neg (t|X \downarrow)^* \land \neg \neg (\forall w \exists sB_{qf}(X, t|X, w, s))^*).$$

By the previous lemma and multiple use of $M^0$, one obtain that

$$\text{EL} + M^0 \vdash \forall X^1 (A(X) \rightarrow t|X \downarrow \land B(X, t|X)).$$

Therefore $\text{EL} + M^0 \vdash \forall X (A(X) \rightarrow \exists YB(X, Y))$. 

□
Combining the theorem with Dorais’s result, we have the following.
Combining the theorem with Dorais’s result, we have the following.

**Proposition.**

There exists a term $t^1$ s.t.

$$\text{RCA} \vdash \forall X^1 (A(X) \rightarrow t|X \downarrow \land B(X, t|X))$$

if and only if

$$\text{EL} + M^0 \vdash \forall X (A(X) \rightarrow \exists Y B(X, Y)),$$

provided that $A(X) \in N_{KM}$ and $B(X, Y)$ is equivalent to some formula $\forall w^\rho \exists s^0 B_{qf}(X, Y, w, s)$ over $\text{EL} + M^0$.

- $N_{KM}$ is the class of formulas defined inductively as;
  - $A_{qf}$ and $\exists x^\rho A_{qf}$ are in $N_{KM}$.
  - If $A_1, A_2$ are in $N_{KM}$, then $A_1 \land A_2, \forall x^\rho A_1$ are in $N_{KM}$.
  - If $A$ is in $N_{KM}$, then $\forall u^\rho \exists v^0 A_{qf} \rightarrow A$ is in $N_{KM}$.
Annoying feature of Intuitionistic systems is lack of the following properties.

- \((A \rightarrow \exists x B) \rightarrow \exists x (A \rightarrow B)\).
- \((\forall x A \rightarrow B) \rightarrow \exists x (A \rightarrow B)\).
On the Syntactical Restriction

- Annoying feature of Intuitionistic systems is lack of the following properties.
  - \((A \rightarrow \exists x B) \rightarrow \exists x (A \rightarrow B)\).
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- However, under the \(M^0\), one can intuitionistically show the followings.
  - \(IP^0(\Pi_1^0, \Sigma_0^0)\):
    \[
    (\forall u^0 A_{qf} \rightarrow \exists x^0 B_{qf}) \rightarrow \exists x^0 (\forall u^0 A_{qf} \rightarrow B_{qf}).
    \]
  - \((\exists x^0 A_{qf} \rightarrow B_{qf}) \rightarrow \exists x^0 (A_{qf} \rightarrow B_{qf}).\)
On the Syntactical Restriction

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- However, under the \(M^0\), one can intuitionistically show the followings.
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  - \((\exists x^0 A_{qf} \rightarrow B_{qf}) \rightarrow \exists x^0 (A_{qf} \rightarrow B_{qf})\).

⇒ Our proposition seems to be applicable to a lot of mathematical statements.
Proposition (Hirst-Mummert 2011, via Modified Realizability Interpretation)

If $E$-$HA^\omega + AC \vdash \forall X^1 (A(X) \rightarrow \exists Y^1 B(X, Y))$, then there exists a term $t^{1\rightarrow 1}$ s.t.

$$E$-$HA^\omega \vdash \forall X (A(X) \rightarrow B(X, tX)),$$

provided that $A(X)$ is existential-free and $B(X, Y) \in \Gamma_1$ where $\Gamma_1$ is the class of formulas defined inductively as:

- $A_{qf}$ is in $\Gamma_1$.
- If $A_1, A_2$ are in $L_K$, then $A_1 \land A_2$, $\forall x A_1$ and $\exists x A_1$ are in $\Gamma_1$.
- If $A_1$ is existential-free and $A_2$ is in $\Gamma_1$, then $A_1 \rightarrow A_2$ is in $\Gamma_1$. 
Corollary.

If $\text{EL} \vdash \forall X (A(X) \rightarrow \exists Y B(X, Y))$, then there exists a term $t^{1 \rightarrow 1}$ of $\text{RCA}^\omega$ s.t.

$$\text{RCA}^\omega \vdash \forall X (A(X) \rightarrow B(X, tX)),$$

provided that $A(X)$ is existential-free and $B(X, Y) \in \Gamma_1$. 
Theorem.

If there exists a term $t^{1 \to 1}$ of RCA$^\omega$ s.t.

$$\text{RCA}^\omega \vdash \forall X (A(X) \to B(X, tX)),$$

then

$$\text{EL} \vdash \forall X (A(X) \to \exists Y B(X, Y)),$$

provided that $A(X)$ is purely universal and $B(X, Y)$ is equivalent to some formula $\forall w^p \exists s^0 B_{qf}(X, Y, w, s)$ over EL.
**Theorem.**

If there exists a term $t^{1\to1}$ of $\text{RCA}^\omega$ s.t.

$$\text{RCA}^\omega \vdash \forall X (A(X) \to B(X, tX)),$$

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provided that $A(X)$ is purely universal and $B(X, Y)$ is equivalent to some formula $\forall w^0 \exists s^0 B_{qf}(X, Y, w, s)$ over EL.

**Proof Sketch.**

As in the previous theorem, by negative translation, we have

$$\text{E-HA}^\omega + \text{QF-AC}^{1,0} + M^0 \vdash \forall X (A(X) \to \exists Y B(X, Y)).$$

By using elimination of extensionality and Dialectica interpretation, we obtain

$$\text{WE-HA}^\omega \vdash \forall X (A(X) \to \exists Y B(X, Y)).$$

The conclusion follows from the conservatively of $\text{WE-HA}^\omega$. □
Combining the theorem with Hirst-Mummert’s result, we have the following.
Combining the theorem with Hirst-Mummert’s result, we have the following.

**Proposition.**

There exists a term $t^{1\rightarrow 1}$ of RCA$^\omega$ s.t.

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\text{RCA}^\omega \vdash \forall X (A(X) \rightarrow B(X, tX))
$$

if and only if

$$
\text{EL} \vdash \forall X (A(X) \rightarrow \exists Y B(X, Y)),
$$

provided that $A(X)$ is purely universal and $B(X, Y)$ is equivalent to some formula $\forall w^\rho \exists s^0 B_{qf}(X, Y, w, s)$ over EL.
Remarks

1. Our two propositions express that in $\omega$ structures, for practical $\Pi_2$ statements, intuitionistic (or constructive recursive) provability is identical with the existence of a uniform algorithm obtaining the witness from the problem and its verification is done in computable mathematics with classical logic.
Remarks

1. Our two propositions express that in $\omega$ structures, for practical $\Pi_2$ statements, intuitionistic (or constructive recursive) provability is identical with the existence of a uniform algorithm obtaining the witness from the problem and its verification is done in computable mathematics with classical logic.

2. One can show the versants of our two propositions where $\text{RCA}^\omega$ and $\text{EL}$ are replaced by $\text{RCA}_0^\omega$ and $\text{EL}_0$ respectively in the same manner. (Note that term $t^{1\rightarrow 1}$ of $\text{RCA}_0^\omega$ is a primitive recursive functional in the sense of Kleene.)
Remarks

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3. All proofs of our propositions are syntactic (just translating formal proofs inductively).
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3. All proofs of our propositions are syntactic (just translating formal proofs inductively).

4. One might obtain this kind of results also for \( \text{RCA} + \text{WKL} \).
References


All of the proof theoretic techniques used for our results are developed in the following books.

Remark. (Yokoyama-F. 2013)

The class $N_K$ for $A$ cannot be extended to involve $\exists u^0 \forall v^0 A_{qf}$ in the previous proposition.

**Proof.**

There is a simple counterexample $B$:

$\forall X \ (X \ is \ finite \rightarrow \exists Y \ s.t. \ its \ upper \ bound \ is \ in \ Y)$. 

$B$ is a statement of form $\forall X \ (\exists u \forall v A_{qf}(X) \rightarrow \exists Y B(X, Y))$ s.t.

- it is provable in EL.
- its **strong** sequential version:

  $\forall \langle X_n \rangle_{n \in \mathbb{N}} \ (\forall n \exists u \forall v A_{qf}(X_n, u, v) \rightarrow \exists \langle Y_n \rangle_{n \in \mathbb{N}} \forall n B(X_n, Y_n))$ implies $ACA$ over $RCA$. 

$\blacksquare$
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There is a simple counterexample $B$:
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  $\forall \langle X_n \rangle_{n \in \mathbb{N}} (\forall n \exists u \forall v A_{qf}(X_n, u, v) \rightarrow \exists \langle Y_n \rangle_{n \in \mathbb{N}} \forall n B(X_n, Y_n))$
  implies ACA over RCA.

Remark: Its weak sequential version:
$\forall \langle X_n \rangle_{n \in \mathbb{N}} \forall \langle u_n \rangle_{n \in \mathbb{N}} (\forall n \forall v A_{qf}(X_n, u_n, v) \rightarrow \exists \langle Y_n \rangle_{n \in \mathbb{N}} \forall n B(X_n, Y_n))$

is trivially provable in RCA.