Weak Lowness Notions for Kolmogorov Complexity

Ian Herbert

National University of Singapore

September 5, 2014

Getting on the Same Page

Definition

A *prefix-free machine* is a partial computable function $M : 2^{<\omega} \to 2^{<\omega}$ such that if $M(\sigma)\downarrow$ then $M(\tau)\uparrow$ for all $\tau \succ \sigma$.

We think of machines as being decoding algorithms.

Definition

The *prefix-free Kolmogorov complexity*, $K(\sigma)$, of a string $\sigma \in 2^{<\omega}$ is the length of the shortest input to the universal prefix-free machine, $\mathbb{U}$, that produces σ.

Definition/Theorem (Schnorr)

A real A is Martin-Löf Random if $\exists b \in \mathbb{N}\ \forall n \in \mathbb{N}$ $K(A\upharpoonright_n) > n - b$.

Getting on the Same Page

Definition

A *prefix-free machine* is a partial computable function $M : 2^{<\omega} \to 2^{<\omega}$ such that if $M(\sigma)\downarrow$ then $M(\tau)\uparrow$ for all $\tau \succ \sigma$.

We think of machines as being decoding algorithms.

Definition

The *prefix-free Kolmogorov complexity*, $K(\sigma)$, of a string $\sigma \in 2^{<\omega}$ is the length of the shortest input to the universal prefix-free machine, $\mathbb{U}$, that produces σ.

Definition/Theorem (Schnorr)

A real A is Martin-Löf Random if $\exists b \in \mathbb{N}\ \forall n \in \mathbb{N}$ $K(A\upharpoonright_n) > n - b$.

Getting on the Same Page

Definition

A *prefix-free machine* is a partial computable function $M : 2^{<\omega} \to 2^{<\omega}$ such that if $M(\sigma)\downarrow$ then $M(\tau)\uparrow$ for all $\tau \succ \sigma$.

We think of machines as being decoding algorithms.

Definition

The *prefix-free Kolmogorov complexity*, $K(\sigma)$, of a string $\sigma \in 2^{<\omega}$ is the length of the shortest input to the universal prefix-free machine, $\mathbb{U}$, that produces σ.

Definition/Theorem (Schnorr)

A real A is Martin-Löf Random if $\exists b \in \mathbb{N}\ \forall n \in \mathbb{N}$ $K(A\restriction_n) > n - b$.

Lowness Notions

Definition (Chaitin; Solovay)

A real A is *K-trivial* if for all n, $K(A\upharpoonright_n) \leq^+ K(n)$.

Definition (Muchnik)

A real A is *low for K* if for all σ, $K(\sigma) \leq^+ K^A(\sigma)$.

Definition (Zambella)

A real A is *low for MLR* if every Martin-Löf random real, Z, is Martin-Löf random relative to A, i.e. $K^A(Z\upharpoonright_n) >^+ n$

Theorem

- *K-trivial* $\Leftrightarrow$ *Low for K* $\Leftrightarrow$ *Low for MLR (Nies 2005).*
- *The K-trivials are closed downward under* $\leq_T$ *(Nies 2005).*
- *The K-trivials are closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003).*
- *There are only countably many K-trivials, and they are all* Δ^0_2 *(Chaitin, 1976).*
- *The K-trivials are all low, and in fact superlow (Nies 2005).*

Theorem

- *K-trivial* $\Leftrightarrow$ *Low for K* $\Leftrightarrow$ *Low for MLR (Nies 2005).*
- *The K-trivials are closed downward under* $\leq_T$ *(Nies 2005).*
- *The K-trivials are closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003).*
- *There are only countably many K-trivials, and they are all* Δ^0_2 *(Chaitin, 1976).*
- *The K-trivials are all low, and in fact superlow (Nies 2005).*

Theorem

- *K-trivial $\Leftrightarrow$ Low for K $\Leftrightarrow$ Low for MLR (Nies 2005).*
- *The K-trivials are closed downward under $\leq_T$ (Nies 2005).*
- *The K-trivials are closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003).*
- *There are only countably many K-trivials, and they are all Δ^0_2 (Chaitin, 1976).*
- *The K-trivials are all low, and in fact superlow (Nies 2005).*

Theorem

- *K-trivial ⇔ Low for K ⇔ Low for MLR (Nies 2005).*
- *The K-trivials are closed downward under $\leq_T$ (Nies 2005).*
- *The K-trivials are closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003).*
- *There are only countably many K-trivials, and they are all Δ^0_2 (Chaitin, 1976).*
- *The K-trivials are all low, and in fact superlow (Nies 2005).*

Theorem

- *K-trivial ⇔ Low for K ⇔ Low for MLR (Nies 2005).*
- *The K-trivials are closed downward under $\leq_T$ (Nies 2005).*
- *The K-trivials are closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003).*
- *There are only countably many K-trivials, and they are all Δ^0_2 (Chaitin, 1976).*
- *The K-trivials are all low, and in fact superlow (Nies 2005).*

One Way to Weaken

Definition

A is Δ^0_2-*bounded K-trivial* if for all n, $K(A\upharpoonright_n) \leq^+ K(n) + f(n)$ for all Δ^0_2 orders f.

Definition

A is Δ^0_2-*bounded low for K* if for all σ, $K(\sigma) \leq^+ K^A(\sigma) + f(\sigma)$ for all Δ^0_2 orders f.

We use $\mathcal{KT}(\Delta^0_2)$ and $\mathcal{LK}(\Delta^0_2)$ to denote these sets of reals. Why Δ^0_2?

Theorem (Baartse, Barmpalias)

There is a Δ^0_3 order g such that $\mathcal{KT}(g)$ is exactly the set of K-trivials.

One Way to Weaken

Definition

A is Δ^0_2-*bounded* K-*trivial* if for all n, $K(A\restriction_n) \leq^+ K(n) + f(n)$ for all Δ^0_2 orders f.

Definition

A is Δ^0_2-*bounded low for* K if for all σ, $K(\sigma) \leq^+ K^A(\sigma) + f(\sigma)$ for all Δ^0_2 orders f.

We use $\mathcal{KT}(\Delta^0_2)$ and $\mathcal{LK}(\Delta^0_2)$ to denote these sets of reals. Why Δ^0_2?

Theorem (Baartse, Barmpalias)

There is a Δ^0_3 order g such that $\mathcal{KT}(g)$ is exactly the set of K-trivials.

One Way to Weaken

Definition

A is Δ^0_2-*bounded K-trivial* if for all n, $K(A\upharpoonright_n) \leq^+ K(n) + f(n)$ for all Δ^0_2 orders f.

Definition

A is Δ^0_2-*bounded low for K* if for all σ, $K(\sigma) \leq^+ K^A(\sigma) + f(\sigma)$ for all Δ^0_2 orders f.

We use $\mathcal{KT}(\Delta^0_2)$ and $\mathcal{LK}(\Delta^0_2)$ to denote these sets of reals. Why Δ^0_2?

Theorem (Baartse, Barmpalias)

There is a Δ^0_3 order g such that $\mathcal{KT}(g)$ is exactly the set of K-trivials.

One Way to Weaken

Definition

A is Δ^0_2-*bounded* K-*trivial* if for all n, $K(A\restriction_n) \leq^+ K(n) + f(n)$ for all Δ^0_2 orders f.

Definition

A is Δ^0_2-*bounded low for* K if for all σ, $K(\sigma) \leq^+ K^A(\sigma) + f(\sigma)$ for all Δ^0_2 orders f.

We use $\mathcal{KT}(\Delta^0_2)$ and $\mathcal{LK}(\Delta^0_2)$ to denote these sets of reals. Why Δ^0_2?

Theorem (Baartse, Barmpalias)

There is a Δ^0_3 order g such that $\mathcal{KT}(g)$ is exactly the set of K-trivials.

Δ^0_2-Bounded Notions

Theorem

- $\mathcal{LK}(\Delta^0_2) \Rightarrow \mathcal{KT}(\Delta^0_2)$, *but the implication does not reverse (H. 2013).*
- $\mathcal{LK}(\Delta^0_2)$ *contains a perfect set. (H. 2013)*
- $\mathcal{LK}(\Delta^0_2)$ *is closed downward under* $\leq_T$, *but for any real* A, *there is a* $B \in \mathcal{KT}(\Delta^0_2)$ *with* $A \leq_T B$. *(H. 2013)*
- $\mathcal{KT}(\Delta^0_2)$ *is closed under effective join, but for any real* A, *there are* $B, C \in \mathcal{LK}(\Delta^0_2)$ *with* $A \leq_T B \oplus C$. *(H.)*

Δ^0_2-Bounded Notions

Theorem

- $\mathcal{LK}(\Delta^0_2) \Rightarrow \mathcal{KT}(\Delta^0_2)$, *but the implication does not reverse (H. 2013).*
- $\mathcal{LK}(\Delta^0_2)$ *contains a perfect set. (H. 2013)*
- $\mathcal{LK}(\Delta^0_2)$ *is closed downward under* $\leq_T$*, but for any real* A*, there is a* $B \in \mathcal{KT}(\Delta^0_2)$ *with* $A \leq_T B$*. (H. 2013)*
- $\mathcal{KT}(\Delta^0_2)$ *is closed under effective join, but for any real* A*, there are* $B, C \in \mathcal{LK}(\Delta^0_2)$ *with* $A \leq_T B \oplus C$*. (H.)*

Δ^0_2-Bounded Notions

Theorem

- $\mathcal{LK}(\Delta^0_2) \Rightarrow \mathcal{KT}(\Delta^0_2)$, *but the implication does not reverse (H. 2013).*
- $\mathcal{LK}(\Delta^0_2)$ *contains a perfect set. (H. 2013)*
- $\mathcal{LK}(\Delta^0_2)$ *is closed downward under* $\leq_T$, *but for any real* A, *there is a* $B \in \mathcal{KT}(\Delta^0_2)$ *with* $A \leq_T B$. *(H. 2013)*
- $\mathcal{KT}(\Delta^0_2)$ *is closed under effective join, but for any real* A, *there are* $B, C \in \mathcal{LK}(\Delta^0_2)$ *with* $A \leq_T B \oplus C$. *(H.)*

Δ^0_2-Bounded Notions

Theorem

- $\mathcal{LK}(\Delta^0_2) \Rightarrow \mathcal{KT}(\Delta^0_2)$, *but the implication does not reverse (H. 2013).*
- $\mathcal{LK}(\Delta^0_2)$ *contains a perfect set. (H. 2013)*
- $\mathcal{LK}(\Delta^0_2)$ *is closed downward under* $\leq_T$, *but for any real* A, *there is a* $B \in \mathcal{KT}(\Delta^0_2)$ *with* $A \leq_T B$. *(H. 2013)*
- $\mathcal{KT}(\Delta^0_2)$ *is closed under effective join, but for any real* A, *there are* $B, C \in \mathcal{LK}(\Delta^0_2)$ *with* $A \leq_T B \oplus C$. *(H.)*

Δ^0_2-Bounded Notions

Proposition

- *No ML-random is in* $\mathcal{LK}(\Delta^0_2)$ *or* $\mathcal{KT}(\Delta^0_2)$.
- *If* A *is* Δ^0_2 *and in* $\mathcal{KT}(\Delta^0_2)$*, then* A *is* K*-trivial.*
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ *Low for Effective Dimension. (Hirshfeldt, Weber)*
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ *Finite Self-Information* $\Rightarrow GL_1$ *(Hirshfeldt, Weber).*
- $\mathcal{KT}(\Delta^0_2)$ *is closed downward under* $\leq_{tt}$ *(H.)*

Δ^0_2-Bounded Notions

Proposition

- *No ML-random is in* $\mathcal{LK}(\Delta^0_2)$ *or* $\mathcal{KT}(\Delta^0_2)$.
- *If A is* Δ^0_2 *and in* $\mathcal{KT}(\Delta^0_2)$, *then A is K-trivial.*
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ *Low for Effective Dimension. (Hirshfeldt, Weber)*
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ *Finite Self-Information* $\Rightarrow GL_1$ *(Hirshfeldt, Weber).*
- $\mathcal{KT}(\Delta^0_2)$ *is closed downward under* $\leq_{tt}$ *(H.)*

Δ^0_2-Bounded Notions

Proposition

- *No ML-random is in* $\mathcal{LK}(\Delta^0_2)$ *or* $\mathcal{KT}(\Delta^0_2)$*.*
- *If* A *is* Δ^0_2 *and in* $\mathcal{KT}(\Delta^0_2)$*, then* A *is* K*-trivial.*
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ *Low for Effective Dimension. (Hirshfeldt, Weber)*
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ *Finite Self-Information* $\Rightarrow GL_1$ *(Hirshfeldt, Weber).*
- $\mathcal{KT}(\Delta^0_2)$ *is closed downward under* $\leq_{tt}$ *(H.)*

Δ^0_2-Bounded Notions

Proposition

- *No ML-random is in* $\mathcal{LK}(\Delta^0_2)$ *or* $\mathcal{KT}(\Delta^0_2)$*.*
- *If* A *is* Δ^0_2 *and in* $\mathcal{KT}(\Delta^0_2)$*, then* A *is* K*-trivial.*
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ *Low for Effective Dimension. (Hirshfeldt, Weber)*
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ *Finite Self-Information* $\Rightarrow GL_1$ *(Hirshfeldt, Weber).*
- $\mathcal{KT}(\Delta^0_2)$ *is closed downward under* $\leq_{tt}$ *(H.)*

Δ^0_2-Bounded Notions

Proposition

- *No ML-random is in* $\mathcal{LK}(\Delta^0_2)$ *or* $\mathcal{KT}(\Delta^0_2)$*.*
- *If* A *is* Δ^0_2 *and in* $\mathcal{KT}(\Delta^0_2)$*, then* A *is* K*-trivial.*
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ *Low for Effective Dimension. (Hirshfeldt, Weber)*
- $\mathcal{LK}(\Delta^0_2) \Rightarrow$ *Finite Self-Information* $\Rightarrow GL_1$ *(Hirshfeldt, Weber).*
- $\mathcal{KT}(\Delta^0_2)$ *is closed downward under* $\leq_{tt}$ *(H.)*

'Strong' reducibilities like $\leq_T$, $\leq_{tt}$, $\leq_m$ have an underlying map: $A \leq B$ iff $\exists \Phi : 2^\omega \to 2^\omega$ with $\Phi(B) = A$.

'Weak' reducibilities do not have such an underlying map. The examples we are concerned with all relate to Kolmogorov complexity.

Weak Reducibilities

Definition (Downey, Hirschfeldt, LaForte)

$A \leq_K B$ iff for all n, $K(A\restriction_n) \leq^+ K(B\restriction_n)$.

Definition (Nies)

$A \leq_{LK} B$ iff for all σ, $K^B(\sigma) \leq^+ K^A(\sigma)$.

Weak Reducibilities

Definition (Downey, Hirschfeldt, LaForte)

$A \leq_K B$ iff for all n, $K(A\restriction_n) \leq^+ K(B\restriction_n)$.

Definition (Nies)

$A \leq_{LK} B$ iff for all σ, $K^B(\sigma) \leq^+ K^A(\sigma)$.

Since we no longer have an underlying map, uncountably many reals may be reducible to a single real under these reducibilities. A natural questions is:

Question

What are the cardinalities of the lower cones for $\mathcal{KT}(\Delta^0_2)$ in $\leq_K$ and $\mathcal{LK}(\Delta^0_2)$ in $\leq_{LK}$?

More Weak Lowness Notions

Definition (Barmpalias, Vlek)

A real A is *infinitely often K-trivial* if for infinitely many n, $K(A\restriction_n) \leq^+ K(n)$.

Definition (Miller)

A real A is *weakly low for K* if for infinitely many σ, $K(\sigma) \leq^+ K^A(\sigma)$.

Infinitely Often K-Triviality

Theorem (Barmpalias, Vlek)

- *Every r.e. set is i.o. K-trivial.*
- *Every $\leq_{tt}$-degree contains an i.o. K-trivial.*
- *There is a perfect set of i.o. K-trivials.*
- *Every set that is computed by a 1-generic is i.o. K-trivial.*
- *No Martin-Löf random set is i.o. K-trivial.*
- *If A is i.o. K-trivial, then A has a countable lower $\leq_K$-cone.*

Infinitely Often K-Triviality

Theorem (Barmpalias, Vlek)

- *Every r.e. set is i.o. K-trivial.*
- *Every $\leq_{tt}$-degree contains an i.o. K-trivial.*
- *There is a perfect set of i.o. K-trivials.*
- *Every set that is computed by a 1-generic is i.o. K-trivial.*
- *No Martin-Löf random set is i.o. K-trivial.*
- *If A is i.o. K-trivial, then A has a countable lower $\leq_K$-cone.*

Theorem (Barmpalias, Vlek)

- *Every r.e. set is i.o. K-trivial.*
- *Every $\leq_{tt}$-degree contains an i.o. K-trivial.*
- *There is a perfect set of i.o. K-trivials.*
- *Every set that is computed by a 1-generic is i.o. K-trivial.*
- *No Martin-Löf random set is i.o. K-trivial.*
- *If A is i.o. K-trivial, then A has a countable lower $\leq_K$-cone.*

Theorem (Barmpalias, Vlek)

- *Every r.e. set is i.o. K-trivial.*
- *Every $\leq_{tt}$-degree contains an i.o. K-trivial.*
- *There is a perfect set of i.o. K-trivials.*
- *Every set that is computed by a 1-generic is i.o. K-trivial.*
- *No Martin-Löf random set is i.o. K-trivial.*
- *If A is i.o. K-trivial, then A has a countable lower $\leq_K$-cone.*

Infinitely Often K-Triviality

Theorem (Barmpalias, Vlek)

- *Every r.e. set is i.o. K-trivial.*
- *Every $\leq_{tt}$-degree contains an i.o. K-trivial.*
- *There is a perfect set of i.o. K-trivials.*
- *Every set that is computed by a 1-generic is i.o. K-trivial.*
- *No Martin-Löf random set is i.o. K-trivial.*
- *If A is i.o. K-trivial, then A has a countable lower $\leq_K$-cone.*

Infinitely Often K-Triviality

Theorem (Barmpalias, Vlek)

- *Every r.e. set is i.o. K-trivial.*
- *Every $\leq_{tt}$-degree contains an i.o. K-trivial.*
- *There is a perfect set of i.o. K-trivials.*
- *Every set that is computed by a 1-generic is i.o. K-trivial.*
- *No Martin-Löf random set is i.o. K-trivial.*
- *If A is i.o. K-trivial, then A has a countable lower $\leq_K$-cone.*

Back to Δ^0_2-Bounded

Theorem (H. with Stephan)

If A is Δ^0_2-bounded K-trivial, then A is infinitely often K-trivial, and this implication does not reverse.

Corollary

Every real in $\mathcal{KT}(\Delta^0_2)$ has a countable lower $\leq_K$-cone.

Back to Δ^0_2-Bounded

Theorem (H. with Stephan)

If A is Δ^0_2-bounded K-trivial, then A is infinitely often K-trivial, and this implication does not reverse.

Corollary

Every real in $\mathcal{KT}(\Delta^0_2)$ has a countable lower $\leq_K$-cone.

Theorem (Miller)

A is weakly low for K iff A is low for Ω, i.e. $\Omega = \mu(dom(\mathbb{U}))$ is ML-random relative to A.

Corollary (via Nies, Stephan, Terwijn)

A is 2-random iff A is ML-random and weakly low for K.

Weak Lowness for K

Theorem (Miller)

A is weakly low for K iff A is low for Ω, i.e. $\Omega = \mu(dom(\mathbb{U}))$ is ML-random relative to A.

Corollary (via Nies, Stephan, Terwijn)

A is 2-random iff A is ML-random and weakly low for K.

Theorem

- *Weakly Low for K is closed downward under $\leq_T$.*
- *If A is weakly low for K then it is GL_1 ($A' \equiv_T A \oplus \emptyset'$) (Nies, Stephan, Terwijn).*
- *If A is Δ^0_2 and weakly low for K, then A is low for K (follows from Hirschfeldt, Nies, Stephan).*

And most importantly for us,

Theorem (Barmpalias, Lewis)

A has a countable lower $\leq_{LK}$-cone if and only if A is weakly low for K.

Weak Lowness for K

Theorem

- *Weakly Low for K is closed downward under $\leq_T$.*
- *If A is weakly low for K then it is GL_1 ($A' \equiv_T A \oplus \emptyset'$) (Nies, Stephan, Terwijn).*
- *If A is Δ^0_2 and weakly low for K, then A is low for K (follows from Hirschfeldt, Nies, Stephan).*

And most importantly for us,

Theorem (Barmpalias, Lewis)

A has a countable lower $\leq_{LK}$-cone if and only if A is weakly low for K.

Weak Lowness for K

Theorem

- *Weakly Low for K is closed downward under $\leq_T$.*
- *If A is weakly low for K then it is GL_1 ($A' \equiv_T A \oplus \emptyset'$) (Nies, Stephan, Terwijn).*
- *If A is Δ^0_2 and weakly low for K, then A is low for K (follows from Hirschfeldt, Nies, Stephan).*

And most importantly for us,

Theorem (Barmpalias, Lewis)

A has a countable lower $\leq_{LK}$-cone if and only if A is weakly low for K.

Weak Lowness for K

Theorem

- *Weakly Low for K is closed downward under $\leq_T$.*
- *If A is weakly low for K then it is GL_1 ($A' \equiv_T A \oplus \emptyset'$) (Nies, Stephan, Terwijn).*
- *If A is Δ^0_2 and weakly low for K, then A is low for K (follows from Hirschfeldt, Nies, Stephan).*

And most importantly for us,

Theorem (Barmpalias, Lewis)

A has a countable lower $\leq_{LK}$-cone if and only if A is weakly low for K.

Back to Δ^0_2-Bounded

So do we have that Δ^0_2-bounded low for K implies weakly low for K, and we can be done?

Unfortunately, no:

Theorem (H.)

Neither of weakly low for K and Δ^0_2-bounded low for K implies the other.

Corollary

Some Δ^0_2-bounded low for K reals have countable lower $\leq_{LK}$-cones, and some have uncountable ones.

Question

Can we characterize those reals that are both Δ^0_2-bounded low for K and weakly low for K?

Back to Δ^0_2-Bounded

So do we have that Δ^0_2-bounded low for K implies weakly low for K, and we can be done?
Unfortunately, no:

Theorem (H.)

Neither of weakly low for K and Δ^0_2-bounded low for K implies the other.

Corollary

Some Δ^0_2-bounded low for K reals have countable lower $\leq_{LK}$-cones, and some have uncountable ones.

Question

Can we characterize those reals that are both Δ^0_2-bounded low for K and weakly low for K?

Back to Δ^0_2-Bounded

So do we have that Δ^0_2-bounded low for K implies weakly low for K, and we can be done?
Unfortunately, no:

Theorem (H.)

Neither of weakly low for K and Δ^0_2-bounded low for K implies the other.

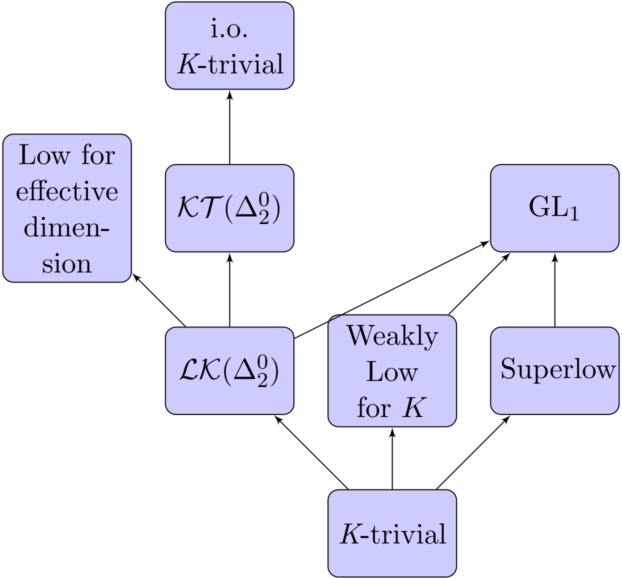
Corollary

Some Δ^0_2-bounded low for K reals have countable lower $\leq_{LK}$-cones, and some have uncountable ones.

Question

Can we characterize those reals that are both Δ^0_2-bounded low for K and weakly low for K?

Picture



i.o. K-trivial
Low for effective dimen-sion
$\mathcal{KT}(\Delta^0_2)$
GL_1
$\mathcal{LK}(\Delta^0_2)$
Weakly Low for K
Superlow
K-trivial

Other Questions

Question

Every nonrecursive weakly low for K set is of hyperimmune degree (Miller, Nies). What about $\mathcal{LK}(\Delta^0_2)$?

Question

What can we say about the internal structures of $\mathcal{LK}(f)$ and $\mathcal{KT}(g)$ for various f and g under $\leq_{LK}$ and $\leq_K$?

Question

What about other lowness notions? C-triviality, lowness for C, etc?

Thanks!