Some RF-type theorems in reverse mathematics

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Introduction

Since H. Friedman started the study of Reverse Mathematics in 1970’s, the relative strength of a lot of mathematical theorems have been investigated in the context of reverse mathematics.

We found that almost all theorems are equivalent to one of the following axioms over the base system, called \( \text{RCA}_0 \):

\[ \text{WKL}_0, \text{ACA}_0, \text{ATR}_0, \Pi^1_1-\text{CA}_0. \]

But recently, some theorems have been found not to be equivalent to any of the above axioms (See The Reverse Mathematics Zoo).

In this talk, we will treat some of such irregular theorems: \textbf{Ramseyan factorization theorem} \((\text{RF}^s_k)\) and its variants.
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But recently, some theorems have been found not to be equivalent to any of the above axioms (See The Reverse Mathematics Zoo).

In this talk, we will treat some of such irregular theorems: **Ramseyan factorization theorem** \((\mathcal{RF}_k^s)\) and its variants.
Contents

1 RF and weak RF (Joint work with T. Yamazaki and K. Yokoyama)

2 A generalization of weak RF

3 “finitary” RF
Contents

1. RF and weak RF  (Joint work with T. Yamazaki and K. Yokoyama)

2. A generalization of weak RF

3. “finitary” RF
**RF and weak RF**

**Definition of RF**

Ramseyan factorization theorem (RF) is a Ramsey-type theorem which is used in automata theory.

It is concerned about
1. infinite sequences and
2. colorings on finite sequences.

**Definition (Ramseyan factorization theorem \((RF^A_B)\))**

For any infinite sequence \(u \in A^\mathbb{N}\) and any coloring on finite sequences \(f : A^{<\mathbb{N}} \to B\), there exists \(v \in (A^{<\mathbb{N}})^\mathbb{N}\) such that

1. \(u = v_0v_1v_2 \ldots\) and
2. \(f(v_iv_{i+1} \ldots v_j) = f(v_{i'}v_{i'+1} \ldots v_{j'})\) for any \(j \geq i > 0\) and \(j' \geq i' > 0\).

\((v_i):\) the \(i\)-th element of \(v\).

We call such \(v\) a Ramseyan factorization for \(u\) and \(f\).
RF and weak RF

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($v_i$: the $i$-th element of $v$.)

We call such $v$ a **Ramseyan factorization** for $u$ and $f$. 
**Example**

Let \( u = 00012112111211112 \ldots \) and 
\[
f : \{0, 1, 2\}^{\mathbb{N}} \to \{0, 1, 2\} \text{ be } f(\sigma) = \text{(the first number of } \sigma)\text{.}
\]
Then \( v = \langle 000, 12, 112, 1112, 11112, \ldots \rangle \) is a R.F. for \( u \) and \( f \).
RF and weak RF

Definition of weak RF

The weak RF (WRF) is the following statement:

**Definition** \((WRF_{A}^{B})\)

For any infinite sequence \(u \in A^{\mathbb{N}}\) and any coloring on finite sequences \(f : A^{<\mathbb{N}} \to B\), there exists \(v \in (A^{<\mathbb{N}})^{\mathbb{N}}\) such that

1. \(u = v_0 v_1 \ldots \) and
2. \(f(v_i) = f(v_{i'})\) for any \(i, i' > 0\).

We call such \(v\) a **weak Ramseyan factorization** for \(u\) and \(f\).

(Fact: Ramseyan factorization \(\Rightarrow\) weak Ramseyan factorization.)
RF and weak RF

Definition of weak RF

The weak RF (WRF) is the following statement:

**Definition (WRF)***

For any infinite sequence $u \in A^\mathbb{N}$ and any coloring on finite sequences $f : A^{< \mathbb{N}} \to B$, there exists $v \in (A^{< \mathbb{N}})^\mathbb{N}$ such that

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We call such $v$ a **weak Ramseyan factorization** for $u$ and $f$.

**(Fact):** Ramseyan factorization $\Rightarrow$ weak Ramseyan factorization.)
RF and weak RF

Relative strength

In a joint work with T. Yamazaki and K. Yokoyama, we showed the following theorems:

Theorem (M./Yamazaki/Yokoyama, 2014)

The following are equivalent over $\text{RCA}_0$:

1. $\text{RT}_2^2$.
2. $\text{RF}_k^N$ $(k \geq 2, \ k \in \omega)$.
3. $\text{RF}_k^n$ $(n, k \geq 2, \ n, k \in \omega)$.

Theorem (M./Yamazaki/Yokoyama, 2014)

$\text{CAC} \Rightarrow \text{WRF}_2^N \Rightarrow \text{ADS}$.
RF and weak RF

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**Theorem (M./Yamazaki/Yokoyama, 2014)**

CAC $\Rightarrow$ WRF$^\mathbb{N}_2$ $\Rightarrow$ ADS.
RF and weak RF

Diagram

RT22

CAC   RFN2   RF22

WRFN2

ADS   WRF22
**RF and weak RF**

Ramsey-type theorem equivalent to $\text{WRF}_k^N$

We also showed the equivalence between $\text{WRF}_k^N$ and a weak version of Ramsey’s theorem:

**Theorem (M./Yamazaki/Yokoyama, 2014)**

The following are equivalent over $\text{RCA}_0$:

1. $\text{psRT}_k^2$.
2. $\text{WRF}_k^N$.

where,

**Definition ($\text{psRT}_k^n$)**

For any coloring $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H = \{a_0 < a_1 < \cdots \}$ such that for any $i, j \in \mathbb{N}$, $P(a_i, a_{i+1}, \ldots, a_{i+n-1}) = P(a_j, a_{j+1}, \ldots, a_{j+n-1})$.

We call such an infinite set $H$ pseudo homogeneous.
RF and weak RF

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We also showed the equivalence between WRF$_k^\mathbb{N}$ and a weak version of Ramsey’s theorem:

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$$P(a_i, a_{i+1}, \ldots, a_{i+n-1}) = P(a_j, a_{j+1}, \ldots, a_{j+n-1}).$$

We call such an infinite set $H$ **pseudo homogeneous**.
RF and weak RF

Open problems

Question 1. We don’t know whether the implications

$$\text{CAC} \Rightarrow \text{WRF}_2^\mathbb{N} \Rightarrow \text{ADS}.$$  

are strict or not.  
(Lerman/Solomon/Towsner recently proved that CAC and ADS are separated.)

Question 2. Is $\text{WRF}_2^2$ strictly weaker than $\text{WRF}_2^\mathbb{N}$?  
(In normal case, $\text{RF}_2^k$ and $\text{RF}_2^{\mathbb{N}}$ are both equivalent to $\text{RT}_2^2$ for any $k \geq 2$.)

Question 3. Does $\text{WRF}_k^n$ imply $\text{WRF}_{k+1}^n$?  
(It is easy to show that $\text{RF}_k^n \Rightarrow \text{RF}_{k+1}^n$ for any $k \geq 2$.)
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\[ \text{CAC} \Rightarrow \text{WRF}_2^N \Rightarrow \text{ADS}. \]

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(In normal case, \( \text{RF}_k^2 \) and \( \text{RF}_k^N \) are both equivalent to \( \text{RT}_2^2 \) for any \( k \geq 2 \).)

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(It is easy to show that \( \text{RF}_k^n \Rightarrow \text{RF}_{k+1}^n \) for any \( k \geq 2 \).)
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3 “finitary” RF
A generalization of weak RF

Definition of \( \leq/-RF \)

Recall Question 2: Is \( \text{WRF}_2^2 \) strictly weaker than \( \text{WRF}_2^N \)?

⇒ A Partial Answer: If \( \text{WRF}_2^2 \) is equivalent to the seemingly little stronger theorem \( \leq 2\text{-RF}_3^2 \), the answer is NO.

Definition (\( \leq/-\text{RF}_B^A \))

For any infinite sequence \( u \in A^N \) and any coloring on finite sequences \( f : A^{<N} \to B \), there exists \( v \in (A^{<N})^N \) such that
1. \( u = v_0v_1 \ldots \) and
2. \( f(v_i; v_{i+1} \ldots v_{i+m-1}) = f(v_j; v_{j+1} \ldots v_{j+n-1}) \) for \( i, j > 0 \) and \( m, n \leq l \).

Remark: \( \text{WRF}_B^A \iff \leq 1\text{-RF}_B^A \).
A generalization of weak RF

Definition of $\leq I\text{-}RF$

Recall **Question 2**: Is $WRF^2_2$ strictly weaker than $WRF^N_2$?

$\Rightarrow$ **A Partial Answer**: If $WRF^2_2$ is equivalent to the seemingly little stronger theorem $\leq 2\text{-}RF^2_3$, the answer is NO.

**Definition ($\leq I\text{-}RF^A_B$)**

For any infinite sequence $u \in A^N$ and any coloring on finite sequences $f : A^{<N} \to B$, there exists $v \in (A^{<N})^N$ such that

1. $u = v_0v_1\ldots$ and
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**Remark**: $WRF^A_B \iff \leq 1\text{-}RF^A_B$. 
A generalization of weak RF

Definition of $\leq l$-RF

Recall **Question 2**: Is $WRF^2_2$ strictly weaker than $WRF^N_2$?

⇒ **A Partial Answer**: If $WRF^2_2$ is equivalent to the seemingly little stronger theorem $\leq 2$-$RF^2_3$, the answer is NO.

**Definition ($\leq l$-$RF^A_B$)**

For any infinite sequence $u \in A^N$ and any coloring on finite sequences
$f : A^{<N} \rightarrow B$, there exists $v \in (A^{<N})^N$ such that
1. $u = v_0 v_1 \ldots$ and
2. $f(v_i; v_{i+1} \ldots v_{i+m-1}) = f(v_j; v_{j+1} \ldots v_{j+n-1})$ for $i, j > 0$ and $m, n \leq l$.

**Remark**: $WRF^A_B \iff \leq 1$-$RF^A_B$. 
A generalization of weak $\text{RF}$

Relative strength

Theorem ($\text{RCA}_0$)

$\leq 2\text{-}\text{RF}_3^2 \Rightarrow \text{WRF}_2^N$. 
A generalization of weak RF

Diagram 2

(Here, $w_{2\text{RF}23}$ denotes $\leq 2-\text{RF}^2_3$, etc.)
A generalization of weak $\text{T}_k$

Ramsey-type theorem equivalent to $\leq \text{RF}_k\mathcal{N}$

We can also get a Ramsey-type theorem equivalent to $\leq \text{RF}_k\mathcal{N}$.

**Definition (space function)**

*For any $X \subseteq \mathbb{N}$, we define a function $\text{space}_X : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$ as follows:*

$$\text{space}_X(\sigma) := |\{x \in X \mid \min \sigma \leq x \leq \max \sigma\}| - \text{lh}(\sigma).$$

**Definition ($<\text{RT}_k^n$)**

*For any coloring $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H$ such that for any $\sigma, \tau \in [H]^n$ satisfying $\text{space}_H(\sigma), \text{space}_H(\tau) < l$, $f(\sigma) = f(\tau)$.*

**Remark:** $\text{psRT}_k^n \iff <1\text{-RT}_k^n$. 
A generalization of weak \( RF \)
Ramsey-type theorem equivalent to \( \leq l-\text{RF}_k^\mathbb{N} \)

**Theorem**

The following are equivalent over \( \text{RCA}_0 \):

1. \( < l-\text{RT}_k^2 \).
2. \( \leq l-\text{RF}_k^\mathbb{N} \).

Recall:

**Theorem (M./Yamazaki/Yokoyama, 2014)**

The following are equivalent over \( \text{RCA}_0 \):

1. \( \text{psRT}_k^2 \).
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A generalization of weak RF
Ramsey-type theorem equivalent to $\leq/l$-$\text{RF}_k^N$

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1. $< l$-$\text{RT}_k^2$.
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Recall:

**Theorem (M./Yamazaki/Yokoyama, 2014)**

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In 2007, T. Tao discussed, on his blog, the relation between “finitary” statements and “infinitary” statements. In his article, he introduced three kinds of “finitary” pigeonhole principles and proved the equivalences between the infinite pigeonhole principle and each of them.

In 2009, J. Gasper and U. Kohlenbach studied the equivalences in the context of reverse mathematics and proved the following theorem:

Theorem (Gasper/Kohlenbach, 2009)

1. $\text{RCA}_0 \vdash \text{FIPP}_2 \rightarrow \text{IPP}$, $\text{RCA}_0 \vdash \text{FIPP}_3 \rightarrow \text{IPP}$.
2. $\text{WKL}_0 \vdash \text{IPP} \rightarrow \text{FIPP}_2$.
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Historical background

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Recently, F. Pelupessy studied its Ramsey version and proved the following theorem:

**Theorem (Pelupessy, 2014)**

1. $\text{RCA}_0 \vdash \text{FRT}^n_k \rightarrow \text{RT}^n_k$.
2. $\text{WKL}_0 \vdash \text{RT}^n_k \rightarrow \text{FRT}^n_k$. 
“finitary” RF
“Finitary” Ramsey’s theorem

Let \( \text{FIN} := \{ \text{the codes of} \} \) all finite subsets of \( \mathbb{N} \).

**Definition (RCA\(_0\))**

\[ F : \text{FIN} \to \mathbb{N} \text{ is asymptotically stable (near infinite sets)} \text{ if for any infinite sequence } X_0 \subseteq X_1 \subseteq \cdots \text{ of finite sets with } X = \bigcup X_i, \exists i \forall j \geq i \ F(X_i) = F(X_j). \]

**Definition (“finitary” infinite Ramsey’s theorem, FRT\(_n^k\))**

\[ \forall F : \text{FIN} \to \mathbb{N}: \text{asymptotically stable} \ \exists R \forall C : [0, R]^d \to k \ \exists H \subseteq [0, R]: C\text{-homogeneous set such that } |H| \geq F(H). \]

**Remark:** \( F(X) := \min X \) is asymptotically stable. Therefore \( \text{PH}_k^n \) is an instance of \( \text{FRT}_k^n \).
Let $\text{FIN} := \{(\text{the codes of}) \text{ all finite subsets of } \mathbb{N}\}$.

**Definition (RCA$_0$)**

$F : \text{FIN} \to \mathbb{N}$ is **asymptotically stable** (near infinite sets) if for any infinite sequence $X_0 \subseteq X_1 \subseteq \cdots$ of finite sets with $X = \bigcup X_i$, $\exists i \forall j \geq i \ F(X_i) = F(X_j)$.

**Definition** ("finitary" infinite Ramsey's theorem, FRT$^n_k$)

$\forall F : \text{FIN} \to \mathbb{N}$: asymptotically stable $\exists R \forall C : [0, R]^d \to k \ \exists H \subseteq [0, R]$: $C$-homogeneous set such that $|H| \geq F(H)$.

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“finitary” RF

“Finitary” Ramsey’s theorem

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**Remark:** $F(X) := \min X$ is asymptotically stable. Therefore $\text{PH}_k^n$ is an instance of $\text{FRT}_k^n$. 
We have several ways to define “finitary” RF.
The point is “How to define the largeness condition $F$ for the Ramseyan factorization $\nu \in (A^{\mathbb{N}})^{\mathbb{N}}$?”

**Case A:** Set $F : (A^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}$ and measure $F(\nu)$.

**Case B:** Set $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ and measure $F(lh(\nu_0), \ldots, lh(\nu_{lh(\nu)-1}))$.

**Case C:** Set $F : A^{\mathbb{N}} \to \mathbb{N}$ and measure $F(\nu_0 \bowtie \cdots \bowtie \nu_{lh(\nu)-1})$. 

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**Case C:** Set $F : A^{\mathbb{N}} \to \mathbb{N}$ and measure $F(\overline{v_0} \cdots \overline{v_{\text{lh}(v)-1}})$.
“finitary” RF

Definition of “finitary” RF

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**Case B:** Set $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ and measure $F(lh(\nu_0), \ldots, lh(\nu_{lh(\nu)-1}))$.

**Case C:** Set $F : A^{\mathbb{N}} \to \mathbb{N}$ and measure $F(\nu_0 \cdots \nu_{lh(\nu)-1})$. 

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Definition of “finitary RF”

**Definition**

\[ F : X^{\leq N} \to \mathbb{N} \text{ is asymptotically stable if for every infinite sequence} \]
\[ \sigma_0 \subseteq \sigma_1 \subseteq \cdots \text{ of } X^{\leq N}, \exists i \forall j \geq i \ F(\sigma_i) = F(\sigma_j). \]

**Definition** (aFR\(_F^A\))

\[ \forall F : (A^{\leq N})^{\leq N} \to \mathbb{N} : \text{a.s. } \exists l \forall f : A^{\leq N} \to B \ \forall u \in A^l \ \exists v \in (A^{\leq N})^{\leq N} \text{ such that } v \text{ is a R.F. for } f \text{ and } u, \text{ and } F(v) \leq \text{lh}(v). \]
Definition of “finitary RF”

Definition

\[ F : X^{<\mathbb{N}} \to \mathbb{N} \] is asymptotically stable if for every infinite sequence \( \sigma_0 \subseteq \sigma_1 \subseteq \cdots \) of \( X^{<\mathbb{N}} \), \( \exists i \forall j \geq i \) \( F(\sigma_i) = F(\sigma_j) \).

Definition \( (a\text{FRF}_B^A) \)

\[ \forall F : (A^{<\mathbb{N}})^{<\mathbb{N}} \to \mathbb{N} : \text{a.s. } \exists l \forall f : A^{<\mathbb{N}} \to B \ \forall u \in A^l \ \exists v \in (A^{<\mathbb{N}})^{<\mathbb{N}} \text{ such that } v \text{ is a R.F. for } f \text{ and } u, \text{ and } F(v) \leq \text{lh}(v) \.]
Definition (bFRF$^A_B$)

\[ \forall F : \mathbb{N}^{< \mathbb{N}} \to \mathbb{N}: \text{a.s. } \exists \forall f : A^{< \mathbb{N}} \to B \ \forall u \in A^I \ \exists v \in (A^{< \mathbb{N}})^{< \mathbb{N}} \text{ such that } v \text{ is a R.F. for } f \text{ and } u, \text{ and } F(\text{lh}(v_0), \ldots, \text{lh}(v_{\text{lh}(v)-1})) \leq \text{lh}(v). \]

Definition (cFRF$^A_B$)

\[ \forall F : A^{< \mathbb{N}} \to \mathbb{N}: \text{a.s. } \exists ! \ \forall f : A^{< \mathbb{N}} \to B \ \forall u \in A^I \ \exists v \in (A^{< \mathbb{N}})^{< \mathbb{N}} \text{ such that } v \text{ is a R.F. for } f \text{ and } u, \text{ and } F(v^*) \leq \text{lh}(v^*). \]

($v^*$ denotes $v_0 \overline{\cdots} v_{\text{lh}(v)-1}$.)
“finitary” RF

Relative strength

Then we can show the following theorems:

**Theorem**

1. \( \text{RCA}_0 \vdash a\text{FRF}^A_B \rightarrow b\text{FRF}^A_B \land c\text{FRF}^A_B \).
2. \( \text{RCA}_0 \vdash b\text{FRF}_2^2 \rightarrow \text{RF}_2^2 \).
3. \( \text{RCA}_0 \vdash c\text{FRF}_2^2 \rightarrow \text{WKL} \).
4. \( \text{WKL}_0 + \text{RF}_2^2 \vdash a\text{FRF}_2^2 \).

**Corollary (RCA\(_0\))**

The following are equivalent:

1. \( \text{WKL} + \text{RT}_2^2 \).
2. \( a\text{FRF}_2^2 \).
3. \( b\text{FRF}_2^2 + c\text{FRF}_2^2 \).
“finitary” RF

Relative strength

Then we can show the following theorems:

Theorem

1. \( \text{RCA}_0 \vdash \text{aFRF}^A_B \rightarrow \text{bFRF}^A_B \land \text{cFRF}^A_B \).
2. \( \text{RCA}_0 \vdash \text{bFRF}_2^2 \rightarrow \text{RF}_2^2 \).
3. \( \text{RCA}_0 \vdash \text{cFRF}_2^2 \rightarrow \text{WKL} \).
4. \( \text{WKL}_0 + \text{RF}_2^2 \vdash \text{aFRF}_2^2 \).

Corollary (RCA\(_0\))

The following are equivalent:

1. \( \text{WKL} + \text{RT}_2^2 \).
2. \( \text{aFRF}_2^2 \).
3. \( \text{bFRF}_2^2 + \text{cFRF}_2^2 \).
“finitary” RF

Diagram 3

RT22 + WKL

\[ \begin{align*}
\text{cFRF22} & \quad \text{bFRF22} & \quad \text{aFRF22} & \quad \text{bFRF22 + cFRF22} \\
& \quad & \quad & \\
\downarrow & \quad & \quad & \\
\text{WKL} & \quad & \text{RT22} & \\
\end{align*} \]
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Thank you.

Thank you for your attention.