Some RF-type theorems in reverse mathematics

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Introduction

Since H. Friedman started the study of Reverse Mathematics in 1970's, the relative strength of a lot of mathematical theorems have been investigated in the context of reverse mathematics.

We found that almost all theorems are equivalent to one of the following axioms over the base system, called RCA_0:

$$\mathsf{WKL}_0,\ \mathsf{ACA}_0,\ \mathsf{ATR}_0,\ \Pi^1_1\text{-}\mathsf{CA}_0.$$

But recently, some theorems have been found not to be equivalent to any of the above axioms (See The Reverse Mathematics Zoo).

In this talk, we will treat some of such irregular theorems: **Ramseyan factorization theorem** (RF^s_k) and its variants.

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RF and weak RF

Definition of RF

Ramseyan factorization theorem (RF) is a Ramsey-type theorem which is used in automata theory.

It is concerned about

1. infinite sequences and
2. colorings on finite sequences.

Definition (Ramseyan factorization theorem ($\mathrm{RF}^{\mathsf{A}}_{\mathsf{B}}$))

For any infinite sequence $u \in A^{\mathbb{N}}$ and any coloring on finite sequences $f : A^{<\mathbb{N}} \to B$, there exists $v \in (A^{<\mathbb{N}})^{\mathbb{N}}$ such that

1. *$u = v_0 v_1 v_2 \dots$ and*
2. *$f(v_i v_{i+1} \dots v_j) = f(v_{i'} v_{i'+1} \dots v_{j'})$ for any $j \geq i > 0$ and $j' \geq i' > 0$.*

(v_i: the i-th element of v.)

We call such v a **Ramseyan factorization** for u and f.

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Example

Let $u = 00012112111211112\ldots$ and
$f : \{0, 1, 2\}^{<\mathbb{N}} \to \{0, 1, 2\}$ be $f(\sigma) =$ (the first number of σ).
Then $v = \langle 000, 12, 112, 1112, 11112, \ldots\rangle$ is a R.F. for u and f.

RF and weak RF

Definition of weak RF

The weak RF (WRF) is the following statement:

Definition (WRF^A_B)

For any infinite sequence $u \in A^{\mathbb{N}}$ and any coloring on finite sequences $f : A^{<\mathbb{N}} \to B$, there exists $v \in (A^{<\mathbb{N}})^{\mathbb{N}}$ such that

1. $u = v_0 v_1 \dots$ *and*
2. $f(v_i) = f(v_{i'})$ *for any* $i, i' > 0$.

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(**Fact**: Ramseyan factorization $\Rightarrow$ weak Ramseyan factorization.)

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RF and weak RF

Relative strength

In a joint work with T. Yamazaki and K. Yokoyama, we showed the following theorems:

Theorem (M./Yamazaki/Yokoyama, 2014)

The following are equivalent over RCA_0*:*

1. RT^2_2.
2. $\mathrm{RF}^{\mathbb{N}}_k \quad (k \geq 2,\ k \in \omega)$.
3. $\mathrm{RF}^n_k \quad (n, k \geq 2,\ n, k \in \omega)$.

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$\mathrm{CAC} \Rightarrow \mathrm{WRF}^{\mathbb{N}}_2 \Rightarrow \mathrm{ADS}$.

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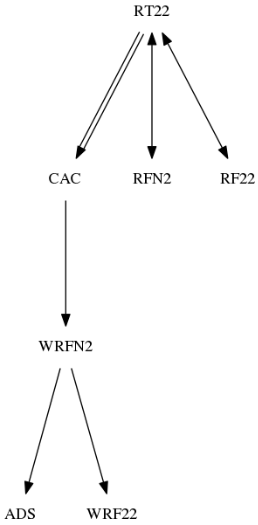

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Ramsey-type theorem equivalent to $\mathrm{WRF}^{\mathbb{N}}_k$

We also showed the equivalence between $\mathrm{WRF}^{\mathbb{N}}_k$ and a weak version of Ramsey's theorem:

Theorem (M./Yamazaki/Yokoyama, 2014)

The following are equivalent over RCA_0*:*

1. psRT^2_k.
2. $\mathrm{WRF}^{\mathbb{N}}_k$.

where,

Definition (psRT^n_k)

For any coloring $P : [\mathbb{N}]^n \to k$*, there exists an infinite set* $H = \{a_0 < a_1 < \cdots\}$ *such that for any* $i, j \in \mathbb{N}$*,* $P(a_i, a_{i+1}, \ldots, a_{i+n-1}) = P(a_j, a_{j+1}, \ldots, a_{j+n-1})$.

We call such an infinite set H **pseudo homogeneous**.

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RF and weak RF

Open problems

Question 1. We don't know whether the implications

$$\mathrm{CAC} \Rightarrow \mathrm{WRF}_2^{\mathbb{N}} \Rightarrow \mathrm{ADS}.$$

are strict or not.
(Lerman/Solomon/Towsner recently proved that CAC and ADS are separated.)

Question 2. Is WRF_2^2 strictly weaker than $\mathrm{WRF}_2^{\mathbb{N}}$?
(In normal case, RF_k^2 and $\mathrm{RF}_k^{\mathbb{N}}$ are both equivalent to RT_2^2 for any $k \geq 2$.)

Question 3. Does WRF_k^n imply WRF_{k+1}^n?
(It is easy to show that $\mathrm{RF}_k^n \Rightarrow \mathrm{RF}_{k+1}^n$ for any $k \geq 2$.)

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A generalization of weak RF

Definition of ${}^{\leq}l$-RF

Recall **Question 2**: Is WRF^2_2 strictly weaker than $\mathrm{WRF}^{\mathbb{N}}_2$?

$\Rightarrow$ **A Partial Answer**: If WRF^2_2 is equivalent to the seemingly little stronger theorem ${}^{\leq}2$-RF^2_3, the answer is NO.

Definition (${}^{\leq}l$-RF^A_B)

For any infinite sequence $u \in A^{\mathbb{N}}$ and any coloring on finite sequences $f : A^{<\mathbb{N}} \to B$, there exists $v \in (A^{<\mathbb{N}})^{\mathbb{N}}$ such that

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Remark: $\mathrm{WRF}^A_B \Leftrightarrow {}^{\leq}1$-$\mathrm{RF}^A_B$.

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A generalization of weak RF

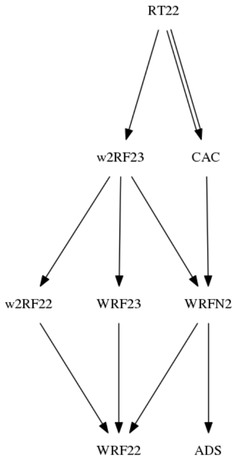
Relative strength

Theorem (RCA_0)

${}^{\leq}2\text{-}\mathrm{RF}^2_3 \Rightarrow \mathrm{WRF}^{\mathbb{N}}_2$.

A generalization of weak RF

Diagram2



(Here, w2RF23 denotes ${}^{\leq}2\text{-}\mathrm{RF}^2_3$, etc.)

A generalization of weak RF

Ramsey-type theorem equivalent to ${}^{\leq}l\text{-}\mathrm{RF}_k^{\mathbb{N}}$

We can also get a Ramsey-type theorem equivalent to ${}^{\leq}l\text{-}\mathrm{RF}_k^{\mathbb{N}}$.

Definition (space function)

For any $X \subseteq \mathbb{N}$, we define a function $\mathrm{space}_X : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$ as follows:

$$\mathrm{space}_X(\sigma) := |\{x \in X \mid \min \sigma \leq x \leq \max \sigma\}| - \mathrm{lh}(\sigma).$$

Definition (${}^{<}l\text{-}\mathrm{RT}_k^n$)

For any coloring $P : [\mathbb{N}]^n \to k$, there exists an infinite set H such that for any $\sigma, \tau \in [H]^n$ satisfying $\mathrm{space}_H(\sigma), \mathrm{space}_H(\tau) < l$, $f(\sigma) = f(\tau)$.

Remark: $\mathrm{psRT}_k^n \Leftrightarrow {}^{<}1\text{-}\mathrm{RT}_k^n$.

A generalization of weak RF

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Theorem

The following are equivalent over RCA_0*:*

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“finitary” RF

Historical background

1. In 2007, T. Tao discussed, on his blog, the relation between “finitary” statements and “infinitary” statements.
   In his article, he introduced three kinds of “finitary” pigeonhole principles and proved the equivalences between the infinite pigeonhole principle and each of them.
2. In 2009, J. Gasper and U. Kohlenbach studied the equivalences in the context of reverse mathematics and proved the following theorem:

Theorem (Gasper/Kohlenbach, 2009)

1. $\mathrm{RCA}_0 \vdash \mathrm{FIPP}_2 \rightarrow \mathrm{IPP}$, $\mathrm{RCA}_0 \vdash \mathrm{FIPP}_3 \rightarrow \mathrm{IPP}$.
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3. Recently, F. Pelupessy studied its Ramsey version and proved the following theorem:

Theorem (Pelupessy, 2014)

1. $\mathsf{RCA}_0 \vdash \mathrm{FRT}^n_k \to \mathrm{RT}^n_k$.
2. $\mathsf{WKL}_0 \vdash \mathrm{RT}^n_k \to \mathrm{FRT}^n_k$.

"finitary" RF

"Finitary" Ramsey's theorem

Let $\mathrm{FIN} := \{$(the codes of) all finite subsets of $\mathbb{N}\}$.

Definition (RCA_0)

$F : \mathrm{FIN} \to \mathbb{N}$ is **asymptotically stable** *(near infinite sets) if for any infinite sequence $X_0 \subseteq X_1 \subseteq \cdots$ of finite sets with $X = \bigcup X_i$, $\exists i \forall j \geq i\ F(X_i) = F(X_j)$.*

Definition ("finitary" infinite Ramsey's theorem, FRT^n_k)

$\forall F : \mathrm{FIN} \to \mathbb{N}$: asymptotically stable $\exists R \forall C : [0, R]^d \to k\ \exists H \subseteq [0, R]$: C-homogeneous set such that $|H| \geq F(H)$.

Remark: $F(X) := \min X$ is asymptotically stable. Therefore PH^n_k is an instance of FRT^n_k.

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Definition of “finitary” RF

We have several ways to define “finitary” RF.
The point is “How to define the largeness condition F for the Ramseyan factorization $v \in (A^{<\mathbb{N}})^{<\mathbb{N}}$?”

Case A: Set $F : (A^{<\mathbb{N}})^{<\mathbb{N}} \to \mathbb{N}$ and measure $F(v)$.

Case B: Set $F : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ and measure $F(\mathrm{lh}(v_0), \dots, \mathrm{lh}(v_{\mathrm{lh}(v)-1}))$.

Case C: Set $F : A^{<\mathbb{N}} \to \mathbb{N}$ and measure $F(v_0^\frown \cdots ^\frown v_{\mathrm{lh}(v)-1})$.

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The point is “How to define the largeness condition F for the Ramseyan factorization $v \in (A^{<\mathbb{N}})^{<\mathbb{N}}$?”

Case A: Set $F : (A^{<\mathbb{N}})^{<\mathbb{N}} \to \mathbb{N}$ and measure $F(v)$.

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“finitary” RF

Definition of “finitary RF

Definition

$F : X^{<\mathbb{N}} \to \mathbb{N}$ *is* **asymptotically stable** *if for every infinite sequence* $\sigma_0 \subseteq \sigma_1 \subseteq \cdots$ *of* $X^{<\mathbb{N}}$, $\exists i \forall j \geq i\ F(\sigma_i) = F(\sigma_j)$.

Definition (aFRF^A_B)

$\forall F : (A^{<\mathbb{N}})^{<\mathbb{N}} \to \mathbb{N}$: *a.s.* $\exists l\ \forall f : A^{<\mathbb{N}} \to B\ \forall u \in A^l\ \exists v \in (A^{<\mathbb{N}})^{<\mathbb{N}}$ *such that* v *is a R.F. for* f *and* u, *and* $F(v) \leq \mathrm{lh}(v)$.

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"finitary" RF

Definition of "finitary RF

Definition (bFRF^A_B)

$\forall F : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$: *a.s.* $\exists l \forall f : A^{<\mathbb{N}} \to B\ \forall u \in A^l\ \exists v \in (A^{<\mathbb{N}})^{<\mathbb{N}}$ *such that* v *is a R.F. for* f *and* u, *and* $F(\mathrm{lh}(v_0), \dots, \mathrm{lh}(v_{\mathrm{lh}(v)-1})) \leq \mathrm{lh}(v)$.

Definition (cFRF^A_B)

$\forall F : A^{<\mathbb{N}} \to \mathbb{N}$: *a.s.* $\exists l\ \forall f : A^{<\mathbb{N}} \to B\ \forall u \in A^l\ \exists v \in (A^{<\mathbb{N}})^{<\mathbb{N}}$ *such that* v *is a R.F. for* f *and* u, *and* $F(v^*) \leq \mathrm{lh}(v^*)$.

(v^* denotes $v_0 ^\frown \cdots ^\frown v_{\mathrm{lh}(v)-1}$.)

“finitary” RF

Relative strength

Then we can show the following theorems:

Theorem

1. $\mathsf{RCA}_0 \vdash \mathrm{aFRF}^A_B \to \mathrm{bFRF}^A_B \wedge \mathrm{cFRF}^A_B$.
2. $\mathsf{RCA}_0 \vdash \mathrm{bFRF}^2_2 \to \mathrm{RF}^2_2$.
3. $\mathsf{RCA}_0 \vdash \mathrm{cFRF}^2_2 \to \mathrm{WKL}$.
4. $\mathsf{WKL}_0 + \mathrm{RF}^2_2 \vdash \mathrm{aFRF}^2_2$.

Corollary (RCA_0)

The following are equivalent:

1. $\mathrm{WKL} + \mathrm{RT}^2_2$.
2. aFRF^2_2.
3. $\mathrm{bFRF}^2_2 + \mathrm{cFRF}^2_2$.

“finitary” RF

Relative strength

Then we can show the following theorems:

Theorem

1. $\mathsf{RCA}_0 \vdash \mathrm{aFRF}^A_B \to \mathrm{bFRF}^A_B \land \mathrm{cFRF}^A_B$.
2. $\mathsf{RCA}_0 \vdash \mathrm{bFRF}^2_2 \to \mathrm{RF}^2_2$.
3. $\mathsf{RCA}_0 \vdash \mathrm{cFRF}^2_2 \to \mathrm{WKL}$.
4. $\mathsf{WKL}_0 + \mathrm{RF}^2_2 \vdash \mathrm{aFRF}^2_2$.

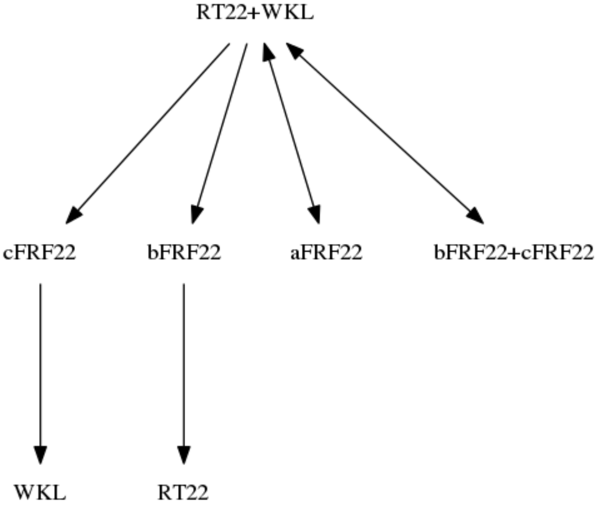
Corollary (RCA_0)

The following are equivalent:

1. $\mathrm{WKL} + \mathrm{RT}^2_2$.
2. aFRF^2_2.
3. $\mathrm{bFRF}^2_2 + \mathrm{cFRF}^2_2$.

"finitary" RF

Diagram3



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Thank you.

Thank you for your attention.