Lindelöf group with non-Lindelöf square and strong negative partition relation

(joint work with Liuzhen Wu)

Yinhe Peng

Chinese Academy of Sciences

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What about the others?

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(a) normality;

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(d) Lindelöf.

It is well-known that for regular spaces, Lindelöf $\Rightarrow$ paracompact $\Rightarrow$ normal & weakly paracompact.

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Weaker version: is the square of hereditarily Lindelöf group normal or weakly paracompact?

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Recall that for regular spaces, Lindelöf $\Rightarrow$ paracompact $\Rightarrow$ normal & weakly paracompact.

Higher finite powers and strong negative partition relations

Next step: for what $n < \omega$ do we have a Lindelöf group (L group) whose n-th power is Lindelöf (L) while its $n+1$-th power is not Lindelöf?

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It was unknown whether there is an L space whose square is an L space.

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Definition

(Strong coloring)([3],[7]) $Pr_0(\kappa, \theta, \sigma)$ *asserts that there is a function* $c : [\kappa]^2 \to \theta$ *such that whenever we are given* $\gamma < \sigma$*, a collection* $\langle a_\alpha : \alpha < \kappa \rangle$ *of pairwise disjoint elements of* $[\kappa]^\gamma$ *and a function* $h : \gamma \times \gamma \to \theta$*, then there are* $\alpha < \beta$ *such that* $c(a_\alpha(i), a_\beta(j)) = h(i, j)$ *for any* $i, j < \gamma$*.*

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It is well-known that $Pr_0(\omega_1, 2, (\omega; 2))$ implies the existence of an L space and Moore in [5] proved that $Pr_0(\omega_1, \omega_1, (\omega; 2))$.

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Now for higher powers, we may first ask whether $Pr_0(\omega_1, 2, (\omega; n))$ is true for larger finite n.

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Our theorem implies that $Pr_0(\omega_1, \omega_1, n)$ for any $n < \omega$ and it is well-known that $Pr_0(\omega_1, \omega_1, \omega)$ is independent of ZFC.

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Theorem (Todorcevic)

$Pr_0(\omega_1, 2, (2; \omega))$ is independent of ZFC

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Theorem (Combinatorial property 1)

For any uncountable families of pairwise disjoint sets $\mathscr{A} \subset [\omega_1]^k$ and $\mathscr{B} \subset [\omega_1]^l$, there are $\mathscr{A}' \in [\mathscr{A}]^{\omega_1}$, $\mathscr{B}' \in [\mathscr{B}]^{\omega_1}$ and $\langle c_{ij} : i < k, j < l\rangle \in \mathbb{Z}^{k\times l}$ such that for any $a \in \mathscr{A}'$, for any $b \in \mathscr{B}'$, if $a < b$, then $osc(a(i), b(j)) = osc(a(i), b(0)) + c_{ij}$ for any $i < k, j < l$.

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$$f(x) = \begin{cases} 0 & : 0 \leq x < \frac{1}{2} \\ 1 & : \frac{1}{2} \leq x < 1. \end{cases}$$

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$c(\alpha, \beta) = f(\{\theta_\alpha osc(\alpha, \beta) + \theta_\beta\})$ witnesses $Pr_0(\omega_1, 2, (2;3))$.

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Denote $\theta_\alpha osc(\alpha, b(i)) + \theta_{b(i)} = a_{\alpha, b(i)}$.

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Going to uncountable subsets, assume $\{x_{\alpha,b}\}$ is always less than $\frac{1}{2}$ (similar for greater than) and moreover $\{x_{\alpha,b}\} \in (\epsilon, \frac{1}{2} - \epsilon)$.

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Going to uncountable subsets, assume $\{x_{\alpha,b}\}$ is always less than $\frac{1}{2}$ (similar for greater than) and moreover $\{x_{\alpha,b}\} \in (\epsilon, \frac{1}{2} - \epsilon)$.

Then when $\{a_{\alpha,b(0)}\}$ goes into all of $(0, \epsilon)$, $(\frac{1}{2} - \epsilon, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2} + \epsilon)$, $(1 - \epsilon, 1)$, $\langle c(\alpha, b(0)), c(\alpha, b(1))\rangle$ will range over $\langle 0, 0\rangle$, $\langle 0, 1\rangle$, $\langle 1, 1\rangle$, $\langle 1, 0\rangle$.

More combinatorial properties of the osc map

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Theorem (Combinatorial property 2)

For any $X \in [\omega_1]^{\omega_1}$*, for any* $k, l < \omega$*, for any* $\langle c_{ij} : i < k, j < l \rangle \in \mathbb{Z}^{k \times l}$ *such that* $c_{i0} = 0$ *for* $i < k$*, there are uncountable families* $\mathscr{A} \subset [X]^k$*,* $\mathscr{B} \subset [X]^l$ *that are pairwise disjoint and for any* $a \in \mathscr{A}$*,* $b \in \mathscr{B}$*, if* $a < b$*, then* $osc(a(i), b(j)) = osc(a(i), b(0)) + c_{ij}$ *for* $i < k, j < l$*.*

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$grp(\mathscr{L})$ – the group generated by $\mathscr{L}$ – is what we need.

An L group with non-Lindelöf square

Theorem

$grp(\mathscr{L})$ is an L group with non-Lindelöf square.

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So none of the 4 properties mentioned above is preserved by taking square for topological groups.

An L ring and an L field

If we use the algebraically closed field $\mathbb{C}$ instead of $\mathbb{R}$, choose appropriate algebraically independent $\{\theta_\alpha : \alpha < \omega_1\}$ and replace *f* by some corresponding function on $\mathbb{C}$ we will get an L ring.

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This construction refuted some conjectures about the relation between one topological space and its square.

(Conjecture 4, [5]) (PFA). If X is a non-separable regular Hausdorff space, then X^2 contains an uncountable discrete subspace.

(Question 17 (a), [4]). Is it consistent that X has a countable network if X^2 has no uncountable discrete subspace?

Construction for general case

For any positive integer n, fix a family of open intervals

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$\{I_{s,\sigma} + s(i)\} \cap \{I_{t,\pi} + t(j)\} = \emptyset$ if $(s, \sigma, i) \neq (t, \pi, j)$.

Construction for general case

For any positive integer n, fix a family of open intervals

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Now define $h : [0, 1) \to \omega$ by

$$h(x) = \begin{cases} \sigma(j) & : \exists s \exists \sigma \exists j \ x \in \{I_{s,\sigma} + s(j)\} \\ 1 & : \textit{otherwise} \end{cases}$$

Construction for general case

Let

$O_\sigma = \cup\{\prod_{j<n}\{(s(j) - \frac{|I_{s,\sigma}|}{4}, s(j) + \frac{|I_{s,\sigma}|}{4})\} : s \in (\mathbb{Q} \cap (0,1))^n$ such that $s(i) \neq s(j)$ for $i \neq j\}$

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Fix $\{\theta_\alpha : \alpha < \omega_1\}$ such that $span^{n\uparrow}\{\theta_\alpha : \alpha < \omega_1\} \subset \bigcap_{\sigma: n \to \omega} \bigcup_{z \in \mathbb{Z}^n} O_\sigma + z$.

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The verification is a generalization of the proof of the L group part.

Go a little further

The above construction provide a witness of $Pr_0(\omega_1, \omega, (\omega; n))$ for any $n < \omega$.

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We can also modify the construction a little bit to get the following:

Theorem

For any $n < \omega$, there is a topological group G such that G^n is an L group and G^{n+1} is neither normal nor weakly paracompact.

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Thank you!