Whitehead's problem and ACA_0

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1. Free group and Whitehead group
2. Whitehead's problem settled in ACA_0
3. Stein's theorem implies ACA_0 over WKL_0
4. Further topics

1. Free group and Whitehead group

Definition

$(G, +, 0)$ is an *Abelian group* if

- $+ : G \times G \to G$ and $0 \in G$.
- $x + (y + z) = (x + y) + z$.
- $x + 0 = 0 + x = x$.
- For all $x \in G$, there exists $y \in G$ such that $x + y = 0$.
- $x + y = y + x$.

"Group" will always mean "abelian group".

Free group

Given abelian group F,

- $B \subset F$ generates F, if every element of F is sum of elements of $B \cup (-B)$, where $-B = \{-x | x \in B\}$.
- $B \subset F$ is independent, if sum of elements of $B \cup (-B)$ is not 0 unless it is the empty sum.
- $B \subset F$ is a basis, if B generates F and B is independent.
- F is free, if it has a basis.

Free group

- The group of integers $\mathbb{Z}$ is free. The direct sums $\mathbb{Z} \oplus \mathbb{Z}$ is free. The infinite direct sum $\bigoplus_{\kappa} \mathbb{Z}$ is also free. In fact, up to isomorphic, these are all free groups.
- So it is obvious that a subgroup of a free group is free.
- For any group P, there is a free group F and a surjective homomorphism $\pi : F \to P$.

Splitting of homomorphism

Given abelian groups F and G, surjective homomorphism $\pi : G \to F$, we say $\rho : F \to G$ is a splitting of π if

- ρ is a homomorphism.
- $\pi\rho$ is the identity map of F.

So if $\pi : G \to F$ has a splitting, then G has a subgroup isomorphic to F. In fact, F is isomorphic to a direct summand of G.

Free group and splitting

- F is free if and only if any surjective homomorphism $\pi : G \to F$ has a splitting.
- F is free if and only if any surjective homomorphism $\pi : G \to F$ with G is free has a splitting.
- Given cardinal $\kappa > 0$, $\Phi_\kappa(F)$ is the sentence: if $\pi : G \to F$ is a surjective homomorphism with kernel isomorphic to $\bigoplus_\kappa \mathbb{Z}$, then π has a splitting.
- If $\kappa > \lambda$, then $\Phi_\kappa(F)$ implies $\Phi_\lambda(F)$.
- F is free if and only if for any $\kappa > 0$, $\Phi_\kappa(F)$ holds.

Whitehead group

A group F is a *Whitehead group* if $\Phi_1(F)$ holds, i.e. any surjective homomorphism $\pi : G \to F$ with kernel isomorphic to $\mathbb{Z}$ has a splitting.

Whitehead's problem

Is there a Whitehead group which is not free?

Stein's theorem

Every countable Whitehead group is free.

Theorem(Shelah)

In L, the universe of constructible sets, every Whitehead group is free.

Theorem(Shelah)

Assume MA_{ω_1}, there is a Whitehead group of cardinality ω_1 which is not free.

2. Whitehead's problem settled in ACA_0

- Concepts "abelian group","basis of a group", "free group", "splitting of a homomorphism", "Whitehead group" are all expressible in second order arithmetic language.
- Whitehead's problem is a problem in second order arithmetic.

Theorem

In ACA_0, Stein's theorem holds, i.e. every Whitehead group is free.

Sketch of the proof

- RCA_0 proves Whitehead group must be torsion-free, i.e. for any $x \in G$ and any $n \neq 0$, $nx \neq 0$.
- RCA_0 proves a subgroup of a Whitehead group is also a Whitehead group.
- In ACA_0, prove if a group satisfies $\forall x \exists n \forall m > n(\frac{x}{m}$ does not exist), then it is free.
- In RCA_0, prove a Whitehead group must satisfy the condition $\forall x \exists n \forall m > n(\frac{x}{m}$ does not exist). This step is a "forcing" essentially. Suppose the condition false for G, then one can construct a homomorphism to G, in the processing of construction, killing all potential splitting.

3. Stein's theorem implies ACA_0 over WKL_0

Theorem

In WKL_0, Stein's theorem implies ACA_0.

Theorem

The followings are equivalent over the theory WKL_0:

- Stein's theorem;
- ACA_0.

Sketch of the proof

- Let $\mathcal{M}$ be a model of $\mathrm{WKL}_0 + \neg \mathrm{ACA}_0$, to find a Whitehead group F in $\mathcal{M}$, but F is not free in $\mathcal{M}$.
- Fix A be r.e. relative to some set in $\mathcal{M}$ and $A \notin \mathcal{M}$. The existence of such A is guaranteed by $\neg \mathrm{ACA}_0$.
- Define group F in $\mathcal{M}$ step by step (using the enumeration of A): we enumerate symbols $x_1, x_2, \ldots$, and if $i \in A$ in step s, then add a symbol x_i' and a relation $2x_i' = x_i$.
- In $\mathcal{M}$, there is no basis of F. Since if there is a basis, then for each i, x_i can be written as a sum of basis elements, so we can judge the odd-even of x_i. So $A \in \mathcal{M}$.

Sketch of the proof

- Using WKL_0, we can prove F is a Whitehead group in $\mathcal{M}$.
- Given $\pi : G \to F$ a surjective homomorphism with kernel generated by y_0.
- For each i, fix p_i such that $\pi(p_i) = x_i$. Use WKL_0 to choose $y_i = p_i$ or $y_i = p_i + y_0$ such that $y_i + y_0$ is odd in G.
- Let us define the suitable y_i': fix p_i' such that $\pi(p_i') = x_i'$. So $\pi(2p_i') = x_i$. So $\pi(2p_i' - y_i) = 0$. So $2p_i' - y_i$ is $2zy_0$ or $(2z+1)y_0$. So $2p_i' - 2zy_0$ is y_i or $y_i + y_0$. Since $y_i + y_0$ is odd, so $2p_i' - 2zy_0 = y_i$. So we can define $y_i' = p_i' - zy_0$.
- Then $\rho(x_i) = y_i$ and $\rho(x_i') = y_i'$ gives us a splitting of π.

4. Further research

Intuition

Whitehead group is free group with base outside our universe.

Goal

To find out some forcing with some "good" properties to add a basis for a Whitehead group.

Thank you.