Expressibility of simple unary generalized quantifier

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Outline

1. Introduction
   - Finite model theory
   - Ehrenfeucht-Frásse game

2. Generalized quantifier
   - Definition
   - Vectorization

3. Expressibility
   - Simple case
   - Other cases
Many theorems in model theory fail if we restrict to *finite structures*.

- **Compactness**
  Let $T = \{ \varphi_{\geq n} \mid n \geq 1 \}$ where $\varphi_{\geq n}$ means "there are at least $n$ elements", then, any finite subset of $T$ is satisfiable in finite structures but $T$ is not.

- **Completeness**

**Theorem (Trakhtenbrot(1950))**

*The halting problem can be reducible to finitely satisfiability problem.* i.e for any TM $M$, we can construct FO-sentence $\varphi_M$ which satisfying:

$M(<M>)$ halts iff $\varphi_M$ is satisfiable by finite structure.
R. Fagin show the first *descriptive complexity* result.

**Theorem (Fagin(1974))**

*Let $K$ be a class of finite structures, then*

\[ K \text{ is } \Sigma_1^1 \text{ definable } \iff K \text{ is NP-computable} \]

**Rmk** $K$ is NP-computable means if finite structure $\mathcal{A}$ is given, then it is NP-computable to decide whether $\mathcal{A} \in K$.

e.g) (undirected) graph $\mathcal{G}$ is 3-colorable iff $\mathcal{G}$ satisfies

\[ \exists C_1 \exists C_2 \exists C_3 ((\forall x (C_1(x) \lor C_2(x) \lor C_3(x))) \land \\
(\forall x \forall y (E(x,y) \rightarrow \land \neg(C_i(x) \land C_i(y)))))) \]
Other complexity classes are also characterized if we restrict to \textit{ordered structures}.

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<tr>
<th>complexity</th>
<th>logic</th>
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<tr>
<td>$\text{AC}^0$</td>
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<td>$\text{FO}+\text{TC operator}(\leq)(\text{Immerman, 83})$</td>
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<td>$\text{FO}+\text{least fixpoint operator}(\leq)$ (Immerman, Vardi 82)</td>
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\[ A \models D_m x \varphi(x) :\iff \#\{ a \in A \mid A \models \varphi(a) \} \equiv 0 \ mod \ m \]

\[ A \models M x \varphi(x) :\iff \#\{ a \in A \mid A \models \varphi(a) \} \geq \#A/2 \]
How can you show "class $K$ is not definable in logic $\mathcal{L}$?"
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→ **Ehrenfeucht-Frásse game** is a tool to show such undefinability.
Let $\tau$ be *finite relational vocabulary*,
$\mathcal{A}, \mathcal{B}$ be $\tau$-str, $k, m \geq 0$, $\bar{a} \in \mathcal{A}^k$, $\bar{b} \in \mathcal{B}^k$
$m$-round EF-game $G_m((\mathcal{A}, \bar{a}), (\mathcal{B}, \bar{b}))$ is defined as follows.

- There are two players (I and II)
- This game consists of $m$-rounds
- $i$-th round (FO-move)
  - I choose $\mathcal{A}$ or $\mathcal{B}$, (assume choose $\mathcal{A}$,) I choose $c_i \in \mathcal{A}$
  - Then II choose $d_i \in \mathcal{B}$ (similarly when I choose $\mathcal{B}$)
- After $m$-th round,
  II win iff $\bar{a}c_1 \cdots c_m \mapsto \bar{b}d_1 \cdots d_m$ is *partial isomorphism*. 
Let’s play $G_2((\mathbb{N}, \leq), (\mathbb{Z}, \leq))$

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For any element \( x \), \( 0x \mapsto a(a - 1) \) is not partial isomorphism.
Let’s play $G_2(\langle \mathbb{N}, \leq \rangle, \langle \mathbb{Z}, \leq \rangle)$

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For any element $x$, $0x \mapsto a(a - 1)$ is not partial isomorphism. In fact, $\mathbb{N} \models \exists x \forall y (x \leq y)$ & $\mathbb{Z} \not\models \exists x \forall y (x \leq y)$
The quantifier rank \( qr(\varphi) \) of FO formula \( \varphi \) is defined as follows.

\[ \varphi: \text{atomic} \Rightarrow qr(\varphi)=0, \quad qr(\neg \varphi)=qr(\varphi), \]

\[ qr(\varphi \lor \psi)=\max\{qr(\varphi), qr(\psi)\}, \quad qr(\exists x \varphi)=qr(\varphi)+1 \]

**Theorem**

The followings are equivalent.

1. \( \text{II has winning strategy in } G_m((\mathcal{A}, \bar{a}), (\mathcal{B}, \bar{b})) \)
2. \( (\mathcal{A}, \bar{a}) \equiv_m (\mathcal{B}, \bar{b}) \)

\[ (\mathcal{A}, \bar{a}) \equiv_m (\mathcal{B}, \bar{b}) :\iff \forall \varphi \ (qr(\varphi) \leq m \Rightarrow \mathcal{A} \models \varphi(\bar{a}) \iff \mathcal{B} \models \varphi(\bar{b})) \]
If we want to show the statement "$K$ is not definable in FO", it’s enough to show

$$\forall n \in \mathbb{N}, \exists A \in K \& \exists B \notin K \text{ s.t } A \equiv_n B$$

Using EF-game, we can show FO can not define the following classes.

- $\{(A, P^A) \mid \#P \equiv 0 \mod m\}$
- $\{(A, \leq) \mid \#A \text{ is even.}\}$
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The expressibility of FO is so limited. We consider to extend FO by adding new quantifier.
First-order formula cannot describe such as 
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Mostowski introduced *generalized quantifier* to express such sentence in 1957.

Lindström extended the concept in 1966, which is also called *Lindström quantifier*. 
Let $\tau := \{R_1, \cdots, R_m\}$ be finite relational vocabulary and $K$ a class of finite $\tau$-str.

**Definition**

generalized quantifier $Q_K$ given by $K$ is defined as follows: for any finite str $\mathcal{A}$,

$$\mathcal{A} \models Q_K \bar{x}_1, \cdots, \bar{x}_m(\varphi_1(\bar{x}_1), \cdots, \varphi_m(\bar{x}_m)) \iff (\mathcal{A}, \varphi_1^{\mathcal{A}}, \cdots, \varphi_m^{\mathcal{A}}) \in K$$

where $\bar{x}_k$ is seq of variables which length is equal to the arity of $R_k$ and $\varphi_k^{\mathcal{A}} := \{\bar{a} \mid \mathcal{A} \models \varphi_k(\bar{a})\}$

We denote the extension of FO equipped with generalized quantifier $Q_K$ by $\text{FO}(Q_K)$.

$Q_K$ is called **simple** if $\tau$ has only one relation symbol and **unary** if $\tau$ has only unary symbols.
Examples

Let \( P, Q \) be unary relation symbols.

- \( K_{\exists} = \{(A, P^A) \mid P^A \neq \emptyset\}, \quad A \models Q_{K_{\exists}} x \varphi(x) \iff \varphi^A \neq 0 \iff A \models \exists x \varphi(x). \)

- \( D_3 = \{(A, P^A) \mid \# P^A \equiv 0 \mod 3\}, \quad A \models Q_{D_3} x \varphi(x) \iff \# \varphi^A \equiv 0 \mod 3 \iff A \models D_3 x \varphi(x). \)

- \( M = \{(A, P^A) \mid \# P^A \geq \# A/2\}, \quad A \models Q_M x \varphi(x) \iff \# \varphi^A \geq \# A/2 \iff A \models M x \varphi(x). \)

- \( I = \{(A, P^A, Q^A) \mid \# P^A = \# Q^A\}, \quad A \models Q_I x, y(\varphi(x), \psi(y)) \iff \# \varphi^A = \# \psi^A. \)
Using generalized quantifiers, we can restate the characterization of some complexity classes.

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Can we also characterize other classes like P or NP in terms of generalized quantifier ??
To capture $P$ in terms of generalized quantifier, we need more definition.

For $\tau = \{R_1, \cdots, R_m\}$, $k > 0$, let $\tau(k) = \{R_1^k, \cdots, R_m^k\}$ where if $R_i$ is $l$-ary relation symbol, $R_i^k$ is $kl$-ary relation symbol.

**Definition**

Let $K$ be a class of $\tau$-str. $k$-th vectorization of $K$ is class of $\tau(k)$-str defined as follows:

$$K^k := \{(A, (R_1^k)^A, \cdots, (R_m^k)^A) | (A^k, (R_1^k)^A, \cdots, (R_m^k)^A) \in K\}$$

**Rmk:** If $(R_i^k)^A$ is $kl$-ary relation over $A$, we can see $(R_i^k)^A$ as $l$-ary relation over $A^k$.

We denote the logic $\text{FO}(\{Q_{k^l} | l > 0\})$ by $\text{FO}^+K$. 
Examples

- \( K_\exists = \{(A, P^A) \mid P^A \neq \emptyset\} \),
  \( \mathcal{A} \models Q_{K_\exists} x_1 x_2 x_3 \varphi(x_1, x_2, x_3) \iff \mathcal{A} \models \exists x_1 \exists x_2 \exists x_3 \varphi(x_1, x_2, x_3) \).

- \( D_3 = \{(A, P^A) \mid \#P^A \equiv 0 \mod 3\} \),
  \( \mathcal{A} \models Q_{D_3^2} x y \varphi(x, y) \iff \#\{(a, b) \in A^2 \mid \mathcal{A} \models \varphi(a, b)\} \equiv 0 \mod 3. \)

- \( M = \{(A, P^A) \mid \#P^A \geq \#A/2\} \),
  \( \mathcal{A} \models Q_{M^2} x y \varphi(x, y) \iff \#\{(a, b) \in A^2 \mid \mathcal{A} \models \varphi(a, b)\} \geq \#A^2/2. \)
We can define a class of finite structures which captures P, i.e.

**Fact**

*There is a class of finite structures $L_P$ s.t. for any class of finite ordered structures $K$, $K$ is $P$-computable iff $K$ is definable in $FO+L_P$. The same statement holds for $L$, $NL$, $NP$, $PSPACE$.*

**Note**

- It is shown that $P$ can’t be captured by the logic $FO(Q_K)$ for any $K$ (Hella, 1992).
- Some classes like $D_m$ collapse *vectorization hierarchy*. i.e. $FO+D_m$ is equivalent to $FO(Q_{D_m})$. 
We investigate expressibility of the most simplest case. Let $\tau = \{P\}$ ($P$: unary), for $S \subseteq \mathbb{N}$, we define a class of $\tau$-str $K_S$ by

$$K_S := \{(A, P^A) \mid \#P^A \in S\}$$

Then, the semantics of the generalized quantifier is given by

$$\mathcal{A} \models Q_{K_S} \varphi(x) \iff \#\varphi^A \in S$$
We investigate expressibility of the most simplest case. Let \( \tau = \{ P \} \) (\( P \): unary), for \( S \subseteq \mathbb{N} \), we define a class of \( \tau \)-str \( K_S \) by

\[
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\]

Then, the semantics of the generalized quantifier is given by

\[
\mathcal{A} \models Q_{K_S}x\varphi(x) \iff \#\varphi^\mathcal{A} \in S
\]

**Question.**

Given two subset \( S, T \subseteq \mathbb{N} \), when is \( \text{FO} + K_T \) (or \( \text{FO}(Q_{K_T}) \)) more expressive than \( \text{FO} + K_S \) (\( \text{FO}(Q_{K_S}) \))??
Definition

For any logic $\mathcal{L}, \mathcal{L}'$, we say $\mathcal{L}'$ is more expressive than $\mathcal{L}$ ($\mathcal{L} \leq \mathcal{L}'$) if for any $\tau$ and any $\tau$-formula $\varphi$ in $\mathcal{L}$, there exists $\tau$-formula $\psi$ in $\mathcal{L}'$ which is equivalent to $\varphi$.

Lemma

For two classes $K, L$,

1. $FO(\text{Q}_K) \leq FO(\text{Q}_L)$ iff $K$ is definable in $FO(\text{Q}_L)$
2. $FO+K \leq FO+L$ iff $K$ is definable in $FO+L$
From now on, $\tau = \{P\}$ ($P$: unary), and $\mathcal{A}$ is $\tau$-str.
Given $S \subseteq \mathbb{N}$, let $S + m := \{n + m \mid n \in S\}$, then for example

$$\mathcal{A} \in K_{S+1} \iff \mathcal{A} \models \exists y (P(y) \land Q_{K_S} x (x \neq y \land P(x)))$$

So, $\text{FO}(Q_{K_{S+1}}) \leq \text{FO}(Q_{K_S})$.

**Theorem (Corredor(1986))**

For $S, T \subseteq \mathbb{N}$,

$$\text{FO}(Q_{K_S}) \leq \text{FO}(Q_{K_T}) \text{ iff } \exists T' \in \mathcal{B}(\{T + m \mid m \geq 0\}) \text{ s.t. } \#(S \Delta T') < \infty$$

**Corollary**

For $m, m' > 0$,

$$\text{FO}(Q_{D_m}) \leq \text{FO}(Q_{D_{m'}}) \text{ iff } m \mid m'$$
(Sketch of proof.) It’s enough to show left to right. 
At first, note that quantifier rank of \( \varphi \in \text{FO}+K_T \) is defined similarly. 
Furthermore, EF-game for FO+K_T is also defined as FO case but add \( Q_{K_T} \)-move:

- I choose \( A \) or \( B \) (assume choose \( A \)), I choose \( X \subseteq A \) which is closed under automorphism which fixes chosen elements ,
- II choose \( Y \subseteq B \) which satisfies \( \#X \in T \iff \#Y \in T \)
- I choose \( b \in Y \), then II choose \( a \in X \).

We assume that \( \forall T' \in \mathcal{B} \{ T + m \mid m \geq 0 \} \) \( \#(S \Delta T') = \infty \), and show for any \( n \in \mathbb{N} \), there exists \( A \in K_S, B \not\in K_S \) s.t 
\( \forall \varphi \in \text{FO}+K_T \) \( \text{qr}(\varphi) \leq n \Rightarrow A \models \varphi \iff B \not\models \varphi \)
we fix $n \in \mathbb{N}$,

Lemma

there exists $u \in S \& v \notin S$ s.t

- $u, v > n$
- for any $m < n$, $u \in T + m$ iff $v \in T + m$

Let $\mathcal{A} = (A, A)$, $\mathcal{B} = (B, B)$ where $\#A = u$, $\#B = v$. Then $\mathcal{A} \in K_S \& \mathcal{B} \notin K_S$. 

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**Lemma**

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Let $\mathcal{A} = (A, A)$, $\mathcal{B} = (B, B)$ where $\#A = u$, $\#B = v$.

Then $\mathcal{A} \in K_S \& \mathcal{B} \notin K_S$.

We need to check II win in EF-game for FO+$K_T$ between $\mathcal{A}$ and $\mathcal{B}$. 
In $i$-th move,

- If I choose FO-move and $a \in A$, II can choose $b \in B$ since $u, v > n$.
- If I choose $Q_{K_T}$-move and $X \subseteq A$,
  - if $X$ does not contain unchosen element, II choose $Y$ as set of correspondings (in this case $\#X = \#Y$).
  - if $X$ contains unchosen element, then $X$ contain all of such elements. II choose $Y$ as set of unchosen elements and correspondings in $X$.
  In this case, $\#X = u - m \& \#Y = v - m$ ($m < n$),

So any case, $\#X \in T$ iff $\#Y \in T$
How about ordered case? For example,

$$\mathcal{A} \in D_4 \iff \mathcal{A} \models Q_{D_2} x P(x) \land Q_{D_2} x (P(x) \land Q_{D_2} y (P(y) \land y \leq x))$$

So, $\text{FO}(Q_{D_4}) \leq \text{FO}(Q_{D_2})$ on ordered.

**Theorem (Nurmonen(2000))**

*For $m, k > 0$, $\text{FO}(Q_{D_{mk}}) \leq \text{FO}(Q_{D_m})$ on ordered.*

**Corollary**

*For $m, m' > 0$,
$\text{FO}(Q_{D_m}) \leq \text{FO}(Q_{D_{m'}})$ on ordered iff $\forall p: \text{prime}, p | m \Rightarrow p | m'$*
How about vectorized case?

\[ A \in K_S \iff A \models \exists z_1 \exists z_2 ((z_1 \neq z_2) \land Q_{K_{2S}}^z xy ((x = z_1 \lor x = z_2) \land P(y))) \]

\[ A \in K_S \iff A \models Q_{K_{S^2}}^z xy (P(x) \land P(y)) \]

where \( 2S := \{2n \mid n \in S\} \), \( S^2 := \{n^2 \mid n \in S\} \).

So, \( \text{FO} + K_S \leq \text{FO} + K_{2S}, \text{FO} + K_{S^2} \)

**Theorem**

*For \( S, T \subseteq \mathbb{N} \),

\( \text{FO} + K_S \leq \text{FO} + K_T \)

iff \( \exists T' \in \mathcal{B} (\{f^{-1}(T) \mid f \in \mathbb{Z}[x]^+\}) \) s.t \( \# (S \Delta T') < \infty \)

\[ f \in \mathbb{Z}[x]^+ \iff f = \sum_{k=0}^{n} a_k x^k \text{ where } a_k \in \mathbb{Z} \& a_n > 0 \]


