Going beyond Peano arithmetic?

Tin Lok Wong

Kurt Gödel Research Center for Mathematical Logic
Vienna, Austria

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First- and second-order arithmetic

- $L_1 = \{0, 1, +, \times, <\}$.
- PA is axiomatized by $\text{PA}^-\text{A}$ and the *induction scheme*

$$\theta(0) \land \forall x (\theta(x) \rightarrow \theta(x + 1)) \rightarrow \forall x \theta(x),$$

where $\theta \in L_1$ possibly with parameters.

- $L_{II} = \{0, 1, +, \times, <, \in\}$ has a *number sort* and a *set sort*.
- The *Big Five* in reverse mathematics are
  - RCA$_0$,
  - WKL$_0$,
  - ACA$_0$,
  - ATR$_0$, and
  - $\Pi^1_1$-CA$_0$,

  in increasing order of strength.
From cuts to second-order arithmetic

- A cut of a model of PA is a nonempty proper initial segment with no maximum.
- No cut is definable in a model of PA.
- We additionally assume all cuts are closed under $\times$.
- Let $I$ is a cut of $M \models PA$. Then

$$\text{Cod}(M/I) = \{X \cap I : X \subseteq M \text{ parametrically definable}\}.$$ 

Notice $(I, \text{Cod}(M/I))$ is an $\mathcal{L}_I$-structure. So we can measure the strength of $I$ against theories in second-order arithmetic using $\text{Th}(I, \text{Cod}(M/I))$. 
Regular and strong cuts

Let $I$ be a cut of a countable $M \models \text{PA}$.

**Theorem (Kirby–Paris 1977)**
The following are equivalent.

(a) $(I, \text{Cod}(M/I)) \models B\Sigma^*_2$.
(b) There is $K \succ M$ of which $I$ is a cut such that $M \setminus I \not\subseteq_{ci} K \setminus I$.

**Theorem (Kirby–Paris 1977)**
The following are equivalent.

(a) $(I, \text{Cod}(M/I)) \models \text{ACA}_0$.
(b) There is $K \succ M$ of which $I$ is a cut such that $M \setminus I \not\subseteq_{ci} K \setminus I$ and $\text{Cod}(M/I) = \text{Cod}(K/I)$.
(c) There is $K \succ M$ of which $I$ is a cut such that $M \setminus I \not\subseteq_{ci} K \setminus I$ and $(\inf_K(M \setminus I), \text{Cod}(K/\inf_K(M \setminus I))) \models \text{RCA}_0$.

$\{x \in K : x < y \text{ for all } y \in M \setminus I\}$
Beyond Peano arithmetic?

Main Question
What are the model-theoretic properties of cuts whose strengths are strictly above PA?

Related research

- Yokoyama found combinatorial characterizations of such cuts.
- Kaye–W and Simpson found (natural?) model-theoretic characterizations of models of ATR\(_0\) and \(\Pi^1_1\)-CA\(_0\).

Approach
Make \((K, J) \succeq (M, I)\) instead of just \(K \supseteq M\).

Definitions

- \(\mathcal{L}_{\text{cut}} = \{0, 1, +, \times, <, \mathbb{I}\}\), where \(\mathbb{I}\) is a unary predicate symbol.
- \(\text{PA}^{\text{cut}} = \text{PA} + \{\mathbb{I} \text{ is a cut closed under } \times\}\).
Elementary extensions \((K, J) \succsim (M, I) \models \text{PA}^\text{cut}\)

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\(I^c = M \setminus I\) and \(J^c = K \setminus J\)

(1) exist by compactness (2) none by Smoryński (3) exist by Smith
Ultra-, amphi-, and contra-regularities

Measure the strength of an $L_{\text{cut}}$-theory $T$ by

$$L_{lll}-\text{Str}(T) = \bigcap \{ \text{Th}(I, \text{Cod}(M/I)) : (M, I) \models T \}.$$ 

**Theorem**

$L_{lll}-\text{Str}(\text{UREG} + \text{AREG} + \text{CREG})$ proves $\text{ACA}_0$ but not $\Delta^1_1-\text{CA}_0$.

**Proof**

There is $M \models \text{PA}$ such that

- $(M, \mathbb{N}) \models \text{UREG} + \text{AREG} + \text{CREG}$; but
- $\text{Cod}(M/\mathbb{N})$ consists precisely of the arithmetic sets.

[Conservative over PA]
The amphiregularity scheme  

*(amphi- means on both sides)*

Scheme for $\text{cf}(\mathbb{I}) < \text{dcf}(\mathbb{I}^c)$

$$\exists \text{cf} \ u \in \mathbb{I} \ \exists y \in \mathbb{I}^c \ \varphi(u, y) \to \exists b \in \mathbb{I}^c \ \exists \text{cf} \ u \in \mathbb{I} \ \exists y > b \ \varphi(u, y)$$

Scheme for $\text{cf}(\mathbb{I}) > \text{dcf}(\mathbb{I}^c)$

$$\exists \text{ci} \ v \in \mathbb{I}^c \ \exists x \in \mathbb{I} \ \psi(v, x) \to \exists a \in \mathbb{I} \ \exists \text{ci} \ v \in \mathbb{I}^c \ \exists x < a \ \psi(v, x)$$

Proposition

The two schemes are equivalent over $\text{PA}^{\text{cut}}$.

Theorem

For a countable $(M, I) \models \text{PA}^{\text{cut}}$, the following are equivalent.

(a) $(M, I) \models \text{AReg}$.
(b) There is $(K, J) \succ (M, I)$ in which $\text{cf}(J) \neq \text{dcf}(K \setminus J)$.
(c) There is $(K, J) \succ (M, I)$ in which $I \subseteq \text{cf} J$ and $M \setminus I \not\subseteq \text{ci} K \setminus J$.
(d) There is $(K, J) \succ (M, I)$ in which $I \not\subseteq \text{cf} J$ and $M \setminus I \subseteq \text{ci} K \setminus J$. 