Going beyond Peano arithmetic?

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First- and second-order arithmetic

- $\mathscr{L}_{\mathrm{I}} = \{0, 1, +, \times, <\}$.
- PA is axiomatized by PA^- and the *induction scheme*

$$\theta(0) \wedge \forall x \, \big(\theta(x) \to \theta(x+1)\big) \to \forall x \, \theta(x),$$

  where $\theta \in \mathscr{L}_{\mathrm{I}}$ possibly with parameters.
- $\mathscr{L}_{\mathrm{II}} = \{0, 1, +, \times, <, \in\}$ has a *number sort* and a *set sort*.
- The *Big Five* in reverse mathematics are
  - RCA_0,
  - WKL_0,
  - ACA_0,
  - ATR_0, and
  - $\Pi^1_1\text{-}\mathsf{CA}_0$,

  in increasing order of strength.

From cuts to second-order arithmetic

- A *cut* of a model of PA is a nonempty proper initial segment with no maximum.
- No cut is definable in a model of PA.
- We additionally assume all cuts are closed under $\times$.
- Let I is a cut of $M \models \mathsf{PA}$. Then

$$\mathrm{Cod}(M/I) = \{X \cap I : X \subseteq M \text{ parametrically definable}\}.$$

Notice $(I, \mathrm{Cod}(M/I))$ is an $\mathscr{L}_{\mathrm{II}}$-structure. So we can

measure the *strength of I* against theories in second-order arithmetic using $\mathrm{Th}(I, \mathrm{Cod}(M/I))$.

Regular and strong cuts

Let I be a cut of a countable $M \models \mathsf{PA}$.

Theorem (Kirby–Paris 1977)

The following are equivalent.

measurable cardinals

(a) $(I, \mathrm{Cod}(M/I)) \models \mathrm{B}\Sigma_2^*$.

(b) There is $K \succcurlyeq M$ of which I is a cut such that $M \setminus I \not\subseteq_{\mathrm{ci}} K \setminus I$.

Theorem (Kirby–Paris 1977)

The following are equivalent.

supercompact cardinals?

(a) $(I, \mathrm{Cod}(M/I)) \models \mathsf{ACA}_0$.

(b) There is $K \succcurlyeq M$ of which I is a cut such that $M \setminus I \not\subseteq_{\mathrm{ci}} K \setminus I$ and $\mathrm{Cod}(M/I) = \mathrm{Cod}(K/I)$.

(c) There is $K \succcurlyeq M$ of which I is a cut such that $M \setminus I \not\subseteq_{\mathrm{ci}} K \setminus I$ and $\big(\inf_K(M \setminus I), \mathrm{Cod}(K/\inf_K(M \setminus I))\big) \models \mathsf{RCA}_0$.

$\{x \in K : x < y \text{ for all } y \in M \setminus I\}$

Beyond Peano arithmetic?

Main Question

What are the model-theoretic properties of cuts whose strengths are *strictly* above PA?

Related research

- Yokoyama found *combinatorial* characterizations of such cuts.
- Kaye–W and Simpson found (natural?) model-theoretic characterizations of *models* of ATR_0 and Π^1_1-CA_0.

Approach

Make $(K, J) \succcurlyeq (M, I)$ instead of just $K \succcurlyeq M$.

Definitions

- $\mathscr{L}_{\mathsf{cut}} = \{0, 1, +, \times, <, \mathbb{I}\}$, where $\mathbb{I}$ is a unary predicate symbol.
- $\mathsf{PA}^{\mathsf{cut}} = \mathsf{PA} + \{\mathbb{I}$ is a cut closed under $\times\}$.

Elementary extensions $(K, J) \succcurlyeq (M, I) \models \mathsf{PA}^{\mathsf{cut}}$

	$J \supsetneq_{\mathsf{e}} I$	$J \not\supseteq_{\mathsf{e}} I$	$J = I$	$J \supsetneq_{\mathsf{cf}} I$	$J \not\supseteq_{\mathsf{cf}} I$
$J^{\mathsf{c}} \supsetneq_{\mathsf{i}} I^{\mathsf{c}}$	(2)	(2)	(2)	(2)	(2)
$J^{\mathsf{c}} \not\supseteq_{\mathsf{i}} I^{\mathsf{c}}$	UREG	ultratall	(3)	ultratall + ultrathick	(1)
$J^{\mathsf{c}} = I^{\mathsf{c}}$	(2)	(2)	exist	(2)	(2)
$J^{\mathsf{c}} \supsetneq_{\mathsf{ci}} I^{\mathsf{c}}$	UREG + AREG	contrathick + ultratall	(3)	contrathick + ultratall + ultrathick	AREG
$J^{\mathsf{c}} \not\supseteq_{\mathsf{ci}} I^{\mathsf{c}}$	UREG + CREG	ultratall	AREG + CREG	AREG + ultratall + ultrathick	(1)

$I^{\mathsf{c}} = M \setminus I$ and $J^{\mathsf{c}} = K \setminus J$

(1) exist by compactness (2) none by Smoryński (3) exist by Smith

Ultra-, amphi-, and contra-regularities

Measure the *strength* of an $\mathscr{L}_{\mathsf{cut}}$-theory T by

$$\mathscr{L}_{\mathbb{II}}\text{-Str}(T) = \bigcap \{ \mathrm{Th}(I, \mathrm{Cod}(M/I)) : (M, I) \models T \}.$$

Theorem

$\mathscr{L}_{\mathbb{II}}$-Str($\mathrm{UReg} + \mathrm{AReg} + \mathrm{CReg}$) proves ACA_0 but not Δ^1_1-CA_0.

conservative over PA

Proof

There is $M \models \mathsf{PA}$ such that

- $(M, \mathbb{N}) \models \mathrm{UReg} + \mathrm{AReg} + \mathrm{CReg}$; but
- $\mathrm{Cod}(M/\mathbb{N})$ consists precisely of the *arithmetic sets*. □

The amphiregularity scheme

(*amphi-* means *on both sides*)

Scheme for $\mathrm{cf}(\mathbb{I}) < \mathrm{dcf}(\mathbb{I}^{\mathrm{c}})$

$$\exists^{\mathrm{cf}} u{\in}\mathbb{I}\ \ \exists y{\in}\mathbb{I}^{\mathrm{c}}\ \ \varphi(u,y) \to \exists b{\in}\mathbb{I}^{\mathrm{c}}\ \ \exists^{\mathrm{cf}} u{\in}\mathbb{I}\ \ \exists y{>}b\ \ \varphi(u,y)$$

Scheme for $\mathrm{cf}(\mathbb{I}) > \mathrm{dcf}(\mathbb{I}^{\mathrm{c}})$

$$\exists^{\mathrm{ci}} v{\in}\mathbb{I}^{\mathrm{c}}\ \ \exists x{\in}\mathbb{I}\ \ \psi(v,x) \to \exists a{\in}\mathbb{I}\ \ \exists^{\mathrm{ci}} v{\in}\mathbb{I}^{\mathrm{c}}\ \ \exists x{<}a\ \ \psi(v,x)$$

AReg

$\varphi, \psi \in \mathscr{L}_{\mathrm{cut}}$

Proposition

The two schemes are equivalent over $\mathrm{PA}^{\mathrm{cut}}$.

Theorem

For a countable $(M, I) \models \mathrm{PA}^{\mathrm{cut}}$, the following are equivalent.

(a) $(M, I) \models \mathrm{AReg}$.

(b) There is $(K, J) \succcurlyeq (M, I)$ in which $\mathrm{cf}(J) \neq \mathrm{dcf}(K \setminus J)$.

(c) There is $(K, J) \succcurlyeq (M, I)$ in which $I \subseteq_{\mathrm{cf}} J$ and $M \setminus I \not\subseteq_{\mathrm{ci}} K \setminus J$.

(d) There is $(K, J) \succcurlyeq (M, I)$ in which $I \not\subseteq_{\mathrm{cf}} J$ and $M \setminus I \subseteq_{\mathrm{ci}} K \setminus J$.