

Dynamic Games with Almost Perfect Information

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Abstract

This paper aims to solve two fundamental problems on finite or infinite horizon dynamic games with complete information. Under some mild conditions, we prove (1) the existence of subgame-perfect equilibria in general dynamic games with simultaneous moves (i.e. almost perfect information), and (2) the existence of *pure-strategy* subgame-perfect equilibria in perfect-information dynamic games with uncertainty. Our results go beyond previous works on continuous dynamic games in the sense that public randomization and the continuity requirement on the state variables are not needed. As an illustrative application, a dynamic stochastic oligopoly market with intertemporally dependent payoffs is considered.

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1 Introduction

Dynamic games with complete information, where players observe the history and move simultaneously or alternatively in finite or infinite horizon, arise naturally in many situations. As noted in [Fudenberg and Tirole \(1991, Chapter 4\)](#) and [Harris, Reny and Robson \(1995\)](#), these games form a general class of regular dynamic games with many applications in economics, political science, and biology. The associated notion of subgame-perfect equilibrium is a fundamental game-theoretic concept. For games with finitely many actions and stages, [Selten \(1965\)](#) showed the existence of subgame-perfect equilibria. The infinite horizon but finite-action case was covered by [Fudenberg and Levine \(1983\)](#).

Since the agents in many economic models need to make continuous choices, it is important to consider dynamic games with general action spaces. However, [Harris, Reny and Robson \(1995\)](#) presented a simple example without subgame-perfect equilibrium, where the game has two players in each of the two stages with only one player having a continuous choice set.¹ Thus, the existence of subgame-perfect equilibria under some suitable conditions remains an open problem even for two-stage dynamic games. The purpose of this paper is to prove the existence of subgame-perfect equilibria for general dynamic games with simultaneous or alternative moves in finite or infinite horizon under some suitable conditions. We shall adopt the general forms of intertemporal preferences, requiring neither stationarity nor additive separability.

For deterministic games with perfect information (i.e., the players move alternatively), the existence of pure-strategy subgame-perfect equilibria was shown in [Harris \(1985\)](#), [Hellwig and Leininger \(1987\)](#), [Börgers \(1989, 1991\)](#) and [Hellwig *et al.* \(1990\)](#) with the model parameters being continuous in actions. However, if the “deterministic” assumption is dropped by introducing a passive player - Nature, then pure-strategy subgame-perfect equilibrium need not exist as shown by a four-stage game in [Harris, Reny and Robson \(1995, p. 538\)](#). In fact, [Luttmer and Mariotti \(2003\)](#) even demonstrated the nonexistence of mixed-strategy subgame-perfect equilibrium in a five-stage game.² Thus, it is still an open problem to show the existence of (pure or mixed-strategy) subgame-perfect equilibria in (finite or infinite horizon) perfect-information dynamic games with uncertainty under some general conditions.

¹As noted in Section 2 of [Harris, Reny and Robson \(1995\)](#), such a dynamic game with continuous choices provides a minimal nontrivial counterexample; see also Exercise 13.4 in [Fudenberg and Tirole \(1991\)](#) for another counterexample of Harris.

²All those counterexamples show that various issues arise when one considers general dynamic games. In the setting with incomplete information, even the equilibrium notion needs to be carefully treated; see [Myerson and Reny \(2015\)](#).

Harris, Reny and Robson (1995) considered continuous dynamic games with almost perfect information in the sense that the players move simultaneously. In such games, there are a finite number of active players and a passive player (Nature), and all the relevant model parameters are assumed to be continuous in both action and state variables (i.e., Nature’s moves). Harris, Reny and Robson (1995) showed the existence of subgame-perfect correlated equilibria by introducing a public randomization device.³ As mentioned above, they also demonstrated the possible nonexistence of subgame-perfect equilibrium through a simple example.

This paper resolves the above two open problems (in both finite and infinite horizon) not just for the class of continuous dynamic games as considered in the earlier works, but more importantly, for general dynamic games in which the relevant model parameters are assumed to be continuous in actions and measurable in states.⁴ In particular, we show the existence of a subgame-perfect equilibrium in a general dynamic game with almost perfect information under some suitable conditions on the state transitions. Theorem 1 (and also Proposition 2) below goes beyond earlier works on continuous dynamic games by dropping public randomization and the continuity requirement on the state variables. Thus, the class of games considered here includes general stochastic games, where the stage payoffs are usually assumed to be continuous in actions and measurable in states.⁵ Our Proposition 1 in Section 2 also presents some regularity properties of the equilibrium payoff correspondences, including compactness and upper hemicontinuity in the action variables.⁶ As an illustrative application of Theorem 1, we consider a dynamic oligopoly market in which firms face stochastic demand/cost and intertemporally dependent payoffs.

We work with the condition that the state transition in each period (except for those periods with one active player) has a component with a suitable density function with respect to some atomless reference measure. This condition is also minimal in the particular sense that the existence result may fail to hold if (1) the passive player, Nature, is not present in the model as shown in Harris, Reny and

³See also Mariotti (2000) and Reny and Robson (2002).

⁴Since the agents need to make optimal choices, the continuity assumption in terms of actions is natural and widely adopted. However, the state variable is not a choice variable, and thus it is unnecessary to impose the state continuity requirement in a general model. Note that the state measurability assumption is the minimal regularity condition one would expect for the model parameters. In addition, we point out that the proof of our main result in the case with state continuity as in Subsection 5.5 is much simpler than that of the general case.

⁵Proposition 2 below does imply a new existence result on subgame-perfect equilibria for a general stochastic game; see Remark 2 in Subsection 4.1.

⁶Such an upper hemicontinuity property in terms of correspondences of equilibrium payoffs, or outcomes, or correlated strategies has been the key for proving the relevant existence results as in Harris (1985), Hellwig and Leininger (1987), Börgers (1989, 1991), Hellwig *et al.* (1990), Harris, Reny and Robson (1995) and Mariotti (2000).

Robson (1995), or (2) with the presence of Nature, the reference measure is not atomless as shown in Luttmer and Mariotti (2003).

For the special class of continuous dynamic games with almost perfect information, we can weaken the atomless reference measure condition slightly. In particular, we simply assume the state transition in each period (except for those periods with one active player) to be an atomless probability measure for any given history, without the requirement of a common reference measure. Note that our model allows the state history to fully influence all the model parameters. Clearly, it covers the case with public randomization in the sense that the state transition has an additional atomless component that is independently and identically distributed across time, and does not enter the payoffs, state transitions and action correspondences. Thus, we obtain the existence result in Harris, Reny and Robson (1995) as a special case.

For dynamic games with almost perfect information, our main result allows the players to take mixed strategies. However, for the special class of dynamic games with perfect information,⁷ we obtain the existence of pure-strategy subgame-perfect equilibria in Corollaries 2 and 3. When Nature is present, the only known general existence result for dynamic games with perfect information is, to the best of our knowledge, for continuous games with public randomization. On the contrary, our Corollary 2 needs neither continuity in the state variables nor public randomization. Furthermore, our Corollary 3 provides a new existence result for continuous dynamic games with perfect information, which generalizes the results of Harris (1985), Hellwig and Leininger (1987), Börgers (1989), and Hellwig *et al.* (1990) to the case when Nature is present.

To prove Theorem 1, we follow the standard three-step procedure in obtaining subgame-perfect equilibria of dynamic games, namely, backward induction, forward induction, and approximation of infinite horizon by finite horizon. Because we drop public randomization and the continuity requirement on the state variables, new technical difficulties arise in each step of the proof. In the step of backward induction, we obtain a new existence result for discontinuous games with stochastic endogenous sharing rules, which extends the main result of Simon and Zame (1990) by allowing the payoff correspondence to be measurable (instead of upper hemicontinuous) in states. For forward induction, we need to obtain strategies that are jointly measurable in history. When there is a public randomization device, the joint measurability follows from the measurable version of Skorokhod's representation theorem and implicit function theorem respectively as in Harris,

⁷Dynamic games with perfect information do have wide applications. For some examples, see Phelps and Pollak (1968) for an intergenerational bequest game, and Peleg and Yaari (1973) and Goldman (1980) for intrapersonal games in which consumers have changing preferences.

Reny and Robson (1995) and Reny and Robson (2002). Here we need to work with the deep “measurable” measurable choice theorem of Mertens (2003). Our main existence result for the finite-horizon case then follows immediately from backward and forward inductions. Lastly, in order to extend the result to the infinite horizon setting, we need to handle various subtle measurability issues due to the lack of continuity in the state variables in our model, which is the most difficult part of the proof for Theorem 1.⁸ As noted in Subsection 5.5 below, the proof for the existence result in the case of continuous dynamic games (i.e., Proposition 3) is much simpler than that of Theorem 1.

The rest of the paper is organized as follows. The model and main result are presented in Section 2. An illustrative application of Theorem 1 to a dynamic oligopoly market with stochastic demand/cost and intertemporally dependent payoffs is given in Section 3. Section 4 provides several variations of the main result. All the proofs are left in the Appendix.

2 Model and main result

In this section, we shall present the model for an infinite-horizon dynamic game with almost perfect information.

The set of players is $I_0 = \{0, 1, \dots, n\}$, where the players in $I = \{1, \dots, n\}$ are active and player 0 is Nature. All players move simultaneously. Time is discrete, and indexed by $t = 0, 1, 2, \dots$

The set of starting points is a closed set $H_0 = X_0 \times S_0$, where X_0 is a compact metric space and S_0 is a Polish space (that is, a complete separable metric space).⁹ At stage $t \geq 1$, player i 's action will be chosen from a subset of a Polish space X_{ti} for each $i \in I$, and $X_t = \prod_{i \in I} X_{ti}$. Nature's action is chosen from a Polish space S_t . Let $X^t = \prod_{0 \leq k \leq t} X_k$ and $S^t = \prod_{0 \leq k \leq t} S_k$. The Borel σ -algebras on X_t and S_t are denoted by $\mathcal{B}(X_t)$ and $\mathcal{B}(S_t)$, respectively. Given $t \geq 0$, a history up to the stage t is a vector

$$h_t = (x_0, s_0, x_1, s_1, \dots, x_t, s_t) \in X^t \times S^t.$$

The set of all such possible histories is denoted by H_t . For any $t \geq 0$, $H_t \subseteq X^t \times S^t$.

For any $t \geq 1$ and $i \in I$, let A_{ti} be a measurable, nonempty and compact valued correspondence from H_{t-1} to X_{ti} such that (1) A_{ti} is sectionally continuous

⁸Because our relevant model parameters are only measurable in the state variables, the usual method of approximating a limit continuous dynamic game by a sequence of finite games, as used in Hellwig *et al.* (1990), Börgers (1991) and Harris, Reny and Robson (1995), is not applicable in our setting.

⁹In each stage $t \geq 1$, there will be a set of action profiles X_t and a set of states S_t . Without loss of generality, we assume that the set of initial points is also a product space for notational consistency.

on X^{t-1} ,¹⁰ and (2) $A_{ti}(h_{t-1})$ is the set of available actions for player $i \in I$ given the history h_{t-1} .¹¹ Let $A_t = \prod_{i \in I} A_{ti}$. Then $H_t = \text{Gr}(A^t) \times S_t$, where $\text{Gr}(A^t)$ is the graph of A^t .

For any $x = (x_0, x_1, \dots) \in X^\infty$, let $x^t = (x_0, \dots, x_t) \in X^t$ be the truncation of x up to the period t . Truncations for $s \in S^\infty$ can be defined similarly. Let H_∞ be the subset of $X^\infty \times S^\infty$ such that $(x, s) \in H_\infty$ if $(x^t, s^t) \in H_t$ for any $t \geq 0$. Then H_∞ is the set of all possible histories in the game.¹²

For any $t \geq 1$, Nature's action is given by a Borel measurable mapping f_{t0} from H_{t-1} to $\mathcal{M}(S_t)$ such that f_{t0} is sectionally continuous on X^{t-1} , where $\mathcal{M}(S_t)$ is endowed with the topology induced by the weak convergence.¹³ For each $t \geq 0$, suppose that λ_t is a Borel probability measure on S_t and λ_t is atomless for $t \geq 1$. Let $\lambda^t = \otimes_{0 \leq k \leq t} \lambda_k$ for $t \geq 0$. We shall assume the following condition on the state transitions.

Assumption 1 (Atomless Reference Measure (ARM)). *A dynamic game is said to satisfy the ‘‘atomless reference measure (ARM)’’ condition if for each $t \geq 1$,*

1. *the probability $f_{t0}(\cdot|h_{t-1})$ is absolutely continuous with respect to λ_t on S_t with the Radon-Nikodym derivative $\varphi_{t0}(h_{t-1}, s_t)$ for all $h_{t-1} \in H_{t-1}$;¹⁴*
2. *the mapping φ_{t0} is Borel measurable and sectionally continuous on X^{t-1} , and integrably bounded in the sense that there is a λ_t -integrable function $\phi_t: S_t \rightarrow \mathbb{R}_+$ such that $\varphi_{t0}(h_{t-1}, s_t) \leq \phi_t(s_t)$ for any $h_{t-1} \in H_{t-1}$ and $s_t \in S_t$.*

For each $i \in I$, the payoff function u_i is a Borel measurable mapping from H_∞ to \mathbb{R}_{++} which is sectionally continuous on X^∞ and bounded by $\gamma > 0$.¹⁵

When one considers a dynamic game with infinite horizon, the following ‘‘continuity at infinity’’ condition is standard.¹⁶ In particular, all discounted repeated games or stochastic games satisfy this condition.

¹⁰Suppose that Y_1, Y_2 and Y_3 are all Polish spaces, and $Z \subseteq Y_1 \times Y_2$. Denote $Z(y_1) = \{y_2 \in Y_2: (y_1, y_2) \in Z\}$ for any $y_1 \in Y_1$. A function (resp. correspondence) $f: Z \rightarrow Y_3$ is said to be sectionally continuous on Y_2 if $f(y_1, \cdot)$ is continuous on $Z(y_1)$ for all y_1 with $Z(y_1) \neq \emptyset$. Similarly, one can define the sectional upper hemicontinuity for a correspondence.

¹¹Suppose that Y and Z are both Polish spaces, and Ψ is a correspondence from Y to Z . Hereafter, the measurability of Ψ , unless specifically indicated, is with respect to the Borel σ -algebra $\mathcal{B}(Y)$ on Y .

¹²A finite horizon dynamic game can be regarded as a special case of an infinite horizon dynamic game in the sense that the action correspondence A_{ti} is point-valued for each player $i \in I$ and $t \geq T$ for some stage $T \geq 1$; see, for example, [Börgers \(1989\)](#) and [Harris, Reny and Robson \(1995\)](#).

¹³For a Polish space A , $\mathcal{M}(A)$ denotes the set of all Borel probability measures on A , and $\Delta(A)$ is the set of all finite Borel measures on A .

¹⁴It is common to have a reference measure when one considers a game with uncountable states. For example, if S_t is a subset of \mathbb{R}^l , then the Lebesgue measure is a natural reference measure.

¹⁵Since u_i is bounded, we can assume that the value of the payoff function is strictly positive without loss of generality.

¹⁶See, for example, [Fudenberg and Levine \(1983\)](#).

For any $T \geq 1$, let

$$w^T = \sup_{\substack{i \in I \\ (x,s) \in H_\infty \\ (\bar{x}, \bar{s}) \in H_\infty \\ x^{T-1} = \bar{x}^{T-1} \\ s^{T-1} = \bar{s}^{T-1}}} |u_i(x, s) - u_i(\bar{x}, \bar{s})|. \quad (1)$$

Assumption 2 (Continuity at Infinity). *A dynamic game is said to be “continuous at infinity” if $w^T \rightarrow 0$ as $T \rightarrow \infty$.*

For player $i \in I$, a strategy f_i is a sequence $\{f_{ti}\}_{t \geq 1}$ such that f_{ti} is a Borel measurable mapping from H_{t-1} to $\mathcal{M}(X_{ti})$ with $f_{ti}(A_{ti}(h_{t-1})|h_{t-1}) = 1$ for all $h_{t-1} \in H_{t-1}$. A strategy profile $f = \{f_i\}_{i \in I}$ is a combination of strategies of all active players.

In any subgame, a strategy combination will generate a probability distribution over the set of possible histories. This probability distribution is called the path induced by the strategy combination in this subgame.

Definition 1. *Suppose that a strategy profile $f = \{f_i\}_{i \in I}$ and a history $h_t \in H_t$ are given for some $t \geq 0$. Let $\tau_t = \delta_{h_t}$, where δ_{h_t} is the probability measure concentrated at the one point h_t . If $\tau_{t'} \in \mathcal{M}(H_{t'})$ has already been defined for some $t' \geq t$, then let*

$$\tau_{t'+1} = \tau_{t'} \diamond (\otimes_{i \in I_0} f_{(t'+1)i}).^{17}$$

Finally, let $\tau \in \mathcal{M}(H_\infty)$ be the unique probability measure on H_∞ such that $\text{Marg}_{H_{t'}} \tau = \tau_{t'}$ for all $t' \geq t$. Then τ is called the path induced by f in the subgame h_t . For all $i \in I$, $\int_{H_\infty} u_i d\tau$ is the payoff of player i in this subgame.

The notion of subgame-perfect equilibrium is defined as follows.

Definition 2 (SPE). *A subgame-perfect equilibrium is a strategy profile f such that for all $i \in I$, $t \geq 0$, and λ^t -almost all $h_t \in H_t$,¹⁸ player i cannot improve his payoff in the subgame h_t by a unilateral change in his strategy.¹⁹*

¹⁷Denote $\otimes_{i \in I_0} f_{(t'+1)i}$ as a transition probability from $H_{t'}$ to $\mathcal{M}(X_{t'+1})$. Notice that the strategy profile is usually represented by a vector. For the notational simplicity later on, we assume that $\otimes_{i \in I_0} f_{(t'+1)i}(\cdot|h_{t'})$ represents the strategy profile in stage $t' + 1$ for a given history $h_{t'} \in H_{t'}$, where $\otimes_{i \in I_0} f_{(t'+1)i}(\cdot|h_{t'})$ is the product of the probability measures $f_{(t'+1)i}(\cdot|h_{t'})$, $i \in I_0$. If λ is a finite measure on X and ν is a transition probability from X to Y , then $\lambda \diamond \nu$ is a measure on $X \times Y$ such that $\lambda \diamond \nu(A \times B) = \int_A \nu(B|x) \lambda(dx)$ for any measurable subsets $A \subseteq X$ and $B \subseteq Y$.

¹⁸A property is said to hold for λ^t -almost all $h_t = (x^t, s^t) \in H_t$ if it is satisfied for λ^t -almost all $s^t \in S^t$ and all $x^t \in H_t(s^t)$.

¹⁹When the state space is uncountable and has a reference measure, it is natural to consider the optimality for almost all sub-histories in the probabilistic sense; see, for example, [Abreu, Pearce and Stacchetti \(1990\)](#) and Footnote 4 therein.

The following theorem is our main result, which shows the existence of a subgame-perfect equilibrium under the conditions of ARM and continuity at infinity. Its proof is left in the appendix.

Theorem 1. *If a dynamic game satisfies the ARM condition and is continuous at infinity, then it possesses a subgame-perfect equilibrium.*

Let $E_t(h_{t-1})$ be the set of subgame-perfect equilibrium payoffs in the subgame h_{t-1} . The following result demonstrates the compactness and upper hemicontinuity properties of the correspondence E_t .

Proposition 1. *If a dynamic game satisfies the ARM condition and is continuous at infinity, then E_t is nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1} .²⁰*

3 Dynamic oligopoly market with sticky prices

In this section, we consider a dynamic oligopoly market in which firms face stochastic demand/cost and intertemporally dependent payoffs. Such a model is a variant of the well-known dynamic oligopoly models as considered in [Green and Porter \(1984\)](#) and [Rotemberg and Saloner \(1986\)](#), which examined the response of firms for demand fluctuations. The key feature of our example is the existence of sticky price effect, which means that the desirability of the good from the demand side could depend on the accumulated past output, and hence gives intertemporally dependent payoff functions.

We consider a dynamic oligopoly market in which n firms produce a homogeneous good in an infinite-horizon setting. The inverse demand function is denoted by $P_t(Q_1, \dots, Q_t, s_t)$, where Q_t is the industry output and s_t the observable demand shock in period t . Notice that the price depends on the past outputs. One possible reason could be that the desirability of consumers will be influenced by their previous consumptions, and hence the price does not adjust instantaneously. We assume that P_t is a bounded function which is continuous in (Q_1, \dots, Q_t) and measurable in s_t . In period t , the shock s_t is selected from the set $S_t = [a_t, b_t]$. We denote firm i 's output in period t by q_{ti} so that $Q_t = \sum_{i=1}^n q_{ti}$. The cost of firm i in period t is $c_{ti}(q_{ti}, s_t)$ given the output q_{ti} and the shock s_t , where c_{ti} is a bounded function continuous in q_{ti} and measurable in s_t . The discount factor of firm i is $\beta_i \in [0, 1)$.

²⁰Suppose that Y_1, Y_2 and Y_3 are all Polish spaces, and $Z \subseteq Y_1 \times Y_2$ and η is a Borel probability measure on Y_1 . Denote $Z(y_1) = \{y_2 \in Y_2 : (y_1, y_2) \in Z\}$ for any $y_1 \in Y_1$. A function (resp. correspondence) $f: Z \rightarrow Y_3$ is said to be essentially sectionally continuous on Y_2 if $f(y_1, \cdot)$ is continuous on $Z(y_1)$ for η -almost all y_1 . Similarly, one can define the essential sectional upper hemicontinuity for a correspondence.

The timing of events is as follows.

1. At the beginning of period t , all firms learn the realization of s_t , which is determined by the law of motion $\kappa_t(\cdot|s_1, Q_1, \dots, s_{t-1}, Q_{t-1})$. Suppose that $\kappa_t(\cdot|s_1, Q_1, \dots, s_{t-1}, Q_{t-1})$ is absolutely continuous with respect to the uniform distribution on S_t with density $\varphi_t(s_1, Q_1, \dots, s_{t-1}, Q_{t-1}, s_t)$, where φ_t is bounded, continuous in (Q_1, \dots, Q_{t-1}) and measurable in (s_1, \dots, s_t) .
2. Firms then simultaneously choose the level of their output $q_t = (q_{t1}, \dots, q_{tn})$, where $q_{ti} \in A_{ti}(s_t, Q_{t-1}) \subseteq \mathbb{R}^l$ for $i = 1, 2, \dots, n$. In particular, the correspondence A_{ti} gives the available actions of firm i , which is nonempty and compact valued, measurable in s_t , and continuous in Q_{t-1} .
3. The strategic choices of all the firms then become common knowledge and this one-period game is repeated.

In period t , given the shock s_t and the output $\{q_k\}_{1 \leq k \leq t}$ up to time t with $q_k = (q_{k1}, \dots, q_{kn})$, the payoff of firm i is

$$u_{ti}(q_1, \dots, q_t, s_t) = \left[P_t \left(\sum_{j=1}^n q_{1j}, \dots, \sum_{j=1}^n q_{tj}, s_t \right) - c_{ti}(q_{ti}, s_t) \right] q_{ti}.$$

Given a sequence of outputs $\{q_t\}_{t \geq 1}$ and shocks $\{s_t\}_{t \geq 1}$, firm i receives the payoff

$$u_{1i}(q_1, s_1) + \sum_{t=2}^{\infty} \beta_i^{t-1} u_{ti}(q_1, \dots, q_t, s_t).$$

Remark 1. *Our dynamic oligopoly model has a non-stationary structure. In particular, the transitions and payoffs are history-dependent. The example captures the scenario that the price of the homogeneous product does not adjust instantaneously to the price indicated by its demand function at the given level of output. For more applications with intertemporally dependent utilities, see, for example, [Ryder and Heal \(1973\)](#), [Fershtman and Kamien \(1987\)](#) and [Becker and Murphy \(1988\)](#). If the model is stationary and the inverse demand function only depends the current output, then the example reduces to be the dynamic oligopoly game with demand fluctuations as considered in [Rotemberg and Saloner \(1986\)](#).*

By condition (1) above, the ARM condition is satisfied. It is also easy to see that the game is continuous at infinity. By Theorem 1, we have the following result.

Corollary 1. *The dynamic oligopoly market possesses a subgame-perfect equilibrium.*

4 Variations of the main result

In this section, we will consider several variations of our main result.

In Subsection 4.1, we still consider dynamic games whose parameters are continuous in actions and measurable in states. We partially relax the ARM condition in two ways. First, we allow the possibility that there is only one active player (but no Nature) at some stages, where the ARM type condition is dropped. Second, we introduce an additional weakly continuous component on the state transitions at any other stages. In addition, we allow the state transition in each period to depend on the current actions as well as on the previous history. Thus, we combine the models for dynamic games with perfect and almost perfect information. We show the existence of a subgame perfect equilibrium such that whenever there is only one active player at some stage, the player can play pure strategy as part of the equilibrium strategies. As a byproduct, we obtain a new existence result for stochastic games. The existence of pure-strategy subgame-perfect equilibria for dynamic games with perfect information (with or without Nature) is provided as an immediate corollary.

In Subsection 4.2, we consider the special case of continuous dynamic games in the sense that all the model parameters are continuous in both action and state variables. We can obtain the corresponding results under a slightly weaker condition. All the previous existence results for continuous dynamic games with perfect and almost perfect information are covered as our special cases.

We will follow the setting and notations in Section 2 as closely as possible. For simplicity, we only describe the changes we need to make on the model. All the proofs are left in the appendix.

4.1 Dynamic games with partially perfect information and a generalized ARM condition

In this subsection, we will generalize the model in Section 2 in three directions. The ARM condition is partially relaxed such that (1) perfect information may be allowed in some stages, and (2) the state transitions have a weakly continuous component in all other stages. In addition, the state transition in any period can depend on the action profile in the current stage as well as on the previous history. The first change allows us to combine the models of dynamic games with perfect and almost perfect information. The second generalization implies that the state transitions need not be norm continuous on the Banach space of finite measures. The last modification covers the model of stochastic games as a special case.

The changes are described below.

1. The state space is a product space of two Polish spaces; that is, $S_t = \hat{S}_t \times \tilde{S}_t$ for each $t \geq 1$.
2. For each $i \in I$, the action correspondence A_{ti} from H_{t-1} to X_{ti} is measurable, nonempty and compact valued, and sectionally continuous on $X^{t-1} \times \hat{S}^{t-1}$. The additional component of Nature is given by a measurable, nonempty and closed valued correspondence \hat{A}_{t0} from $\text{Gr}(A_t)$ to \hat{S}^t , which is sectionally continuous on $X^t \times \hat{S}^{t-1}$. Then $H_t = \text{Gr}(\hat{A}_{t0}) \times \tilde{S}_t$, and H_∞ is the subset of $X^\infty \times S^\infty$ such that $(x, s) \in H_\infty$ if $(x^t, s^t) \in H_t$ for any $t \geq 0$.
3. The choices of Nature depend not only on the history h_{t-1} , but also on the action profile x_t in the current stage. The state transition $f_{t0}(h_{t-1}, x_t) = \hat{f}_{t0}(h_{t-1}, x_t) \diamond \tilde{f}_{t0}(h_{t-1}, x_t)$, where \hat{f}_{t0} is a transition probability from $\text{Gr}(A_t)$ to $\mathcal{M}(\hat{S}_t)$ such that $\hat{f}_{t0}(\hat{A}_{t0}(h_{t-1}, x_t) | h_{t-1}, x_t) = 1$ for all $(h_{t-1}, x_t) \in \text{Gr}(A_t)$, and \tilde{f}_{t0} is a transition probability from $\text{Gr}(\hat{A}_{t0})$ to $\mathcal{M}(\tilde{S}_t)$.
4. For each $i \in I$, the payoff function u_i is a Borel measurable mapping from H_∞ to \mathbb{R}_{++} which is bounded by $\gamma > 0$, and sectionally continuous on $X^\infty \times \hat{S}^\infty$.

We allow the possibility for the players to have perfect information in some stages. For $t \geq 1$, let

$$N_t = \begin{cases} 1, & \text{if } f_{t0}(h_{t-1}, x_t) \equiv \delta_{s_t} \text{ for some } s_t \text{ and} \\ & |\{i \in I : A_{ti} \text{ is not point valued}\}| = 1; \\ 0, & \text{otherwise,} \end{cases}$$

where $|K|$ represents the number of points in the set K . Thus, if $N_t = 1$ for some stage t , then the player who is active in the period t is the only active player and has perfect information.

We will drop the ARM condition in those periods with only one active player, and weaken the ARM condition in other periods.

Assumption 3 (ARM'). 1. For any $t \geq 1$ with $N_t = 1$, S_t is a singleton set $\{\acute{s}_t\}$ and $\lambda_t = \delta_{\acute{s}_t}$.

2. For each $t \geq 1$ with $N_t = 0$, \hat{f}_{t0} is sectionally continuous on $X^t \times \hat{S}^{t-1}$, where the range space $\mathcal{M}(\hat{S}_t)$ is endowed with topology of weak convergence of measures on \hat{S}_t . The probability measure $\tilde{f}_{t0}(\cdot | h_{t-1}, x_t, \hat{s}_t)$ is absolutely continuous with respect to an atomless Borel probability measure λ_t on \tilde{S}_t for all $(h_{t-1}, x_t, \hat{s}_t) \in \text{Gr}(\hat{A}_{t0})$, and $\varphi_{t0}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t)$ is the corresponding density.²¹

²¹In this subsection, a property is said to hold for λ^t -almost all $h_t \in H_t$ if it is satisfied for λ^t -almost all $\tilde{s}^t \in \tilde{S}^t$ and all $(x^t, \hat{s}^t) \in H_t(\tilde{s}^t)$.

3. The mapping φ_{t0} is Borel measurable and sectionally continuous on $X^t \times \hat{S}^t$, and integrably bounded in the sense that there is a λ_t -integrable function $\phi_t: \tilde{S}_t \rightarrow \mathbb{R}_+$ such that $\varphi_{t0}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t) \leq \phi_t(\tilde{s}_t)$ for any $(h_{t-1}, x_t, \hat{s}_t)$.

The following proposition shows that the existence result is still true in this more general setting.

Proposition 2. *If an infinite-horizon dynamic game satisfies the ARM condition and is continuous at infinity, then it possesses a subgame-perfect equilibrium f . In particular, for $j \in I$ and $t \geq 1$ such that $N_t = 1$ and player j is the only active player in this period, f_{tj} can be deterministic. Furthermore, the equilibrium payoff correspondence E_t is nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^{t-1} \times \hat{S}^{t-1}$.*

Remark 2. *The proposition above also implies a new existence result of subgame-perfect equilibria for stochastic games. Consider a standard stochastic game with uncountable states as in [Mertens and Parthasarathy \(2003\)](#). [Mertens and Parthasarathy \(2003\)](#) proved the existence of a subgame-perfect equilibrium by assuming the state transitions to be norm continuous with respect to the actions in the previous stage. On the contrary, our [Proposition 2](#) allows the state transitions to have a weakly continuous component.*

Dynamic games with perfect information is a special class of dynamic games in which players move sequentially. As noted in [Footnote 7](#), such games have been extensively studied and found wide applications in economics. As an immediate corollary, an equilibrium existence result for dynamic games with perfect information is given below.

Corollary 2. *If a dynamic game with perfect information satisfies the ARM condition and is continuous at infinity, then it possesses a pure-strategy subgame-perfect equilibrium.*

4.2 Continuous dynamic games with partially perfect information

In this subsection, we will study an infinite-horizon dynamic game with a continuous structure. As in the previous subsection, we allow the state transition to depend on the action profile in the current stage as well as on the previous history, and the players may have perfect information in some stages.

1. For each $t \geq 1$, the choices of Nature depends not only on the history h_{t-1} , but also on the action profile x_t in this stage. For any $t \geq 1$, suppose that A_{t0} is a

continuous, nonempty and closed valued correspondence from $\text{Gr}(A_t)$ to S_t . Then $H_t = \text{Gr}(A_{t0})$, and H_∞ is the subset of $X^\infty \times S^\infty$ such that $(x, s) \in H_\infty$ if $(x^t, s^t) \in H_t$ for any $t \geq 0$.

2. Nature's action is given by a mapping f_{t0} from $\text{Gr}(A_t)$ to $\mathcal{M}(S_t)$ such that $f_{t0}(A_{t0}(h_{t-1}, x_t) | h_{t-1}, x_t) = 1$ for all $(h_{t-1}, x_t) \in \text{Gr}(A_t)$.
3. For each $t \geq 1$, let

$$N_t = \begin{cases} 1, & \text{if } f_{t0}(h_{t-1}, x_t) \equiv \delta_{s_t} \text{ for some } s_t \text{ and} \\ & |\{i \in I : A_{ti} \text{ is not point valued}\}| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3. A dynamic game is said to be continuous if for each t and i ,

1. the action correspondence A_{ti} is continuous on H_{t-1} ;
2. the transition probability f_{t0} is a continuous mapping from $\text{Gr}(A_t)$ to $\mathcal{M}(S_t)$, where $\mathcal{M}(S_t)$ is endowed with topology of weak convergence of measures on S_t ;
3. the payoff function u_i is continuous on H_∞ .

Note that the ‘‘continuity at infinity’’ condition is automatically satisfied in a continuous dynamic game.

Next, we propose the condition of ‘‘atomless transitions’’ on the state space, which means that the state transition is an atomless probability measure in any stage. This condition is slightly weaker than the ARM condition.

- Assumption 4** (Atomless Transitions).
 1. For any $t \geq 1$ with $N_t = 1$, S_t is a singleton set $\{s_t\}$.
 2. For each $t \geq 1$ with $N_t = 0$, $f_{t0}(h_{t-1})$ is an atomless Borel probability measure for each $h_{t-1} \in H_{t-1}$.

Since we work with continuous dynamic games, we can adopt a slightly stronger notion of subgame-perfect equilibrium. That is, each player's strategy is optimal in every subgame given the strategies of all other players.

Definition 4 (SPE'). A subgame-perfect equilibrium is a strategy profile f such that for all $i \in I$, $t \geq 0$, and all $h_t \in H_t$, player i cannot improve his payoff in the subgame h_t by a unilateral change in his strategy.

The result on the equilibrium existence is presented below.

Proposition 3. *If a continuous dynamic game has atomless transitions, then it possesses a subgame-perfect equilibrium f . In particular, for $j \in I$ and $t \geq 1$ such that $N_t = 1$ and player j is the only active player in this period, f_{tj} can be deterministic. In addition, E_t is nonempty and compact valued, and upper hemicontinuous on H_{t-1} for any $t \geq 1$.*

Remark 3. *Proposition 3 goes beyond the main result of [Harris, Reny and Robson \(1995\)](#). They proved the existence of a subgame-perfect correlated equilibrium in a continuous dynamic game with almost perfect information by introducing a public randomization device, which does not influence the payoffs, transitions or action correspondences. It is easy to see that their model automatically satisfies the condition of atomless transitions. The state in our model is completely endogenous in the sense that it affects all the model parameters such as payoffs, transitions, and action correspondences.*

Remark 4. *Proposition 3 above provides a new existence result for continuous stochastic games. As remarked in the previous subsection, the existence of subgame-perfect equilibria has been proved for general stochastic games with a stronger continuity assumption on the state transitions, namely the norm continuity. On the contrary, we only need to require the state transitions to be weakly continuous.*

Remark 5. *The condition of atomless transitions is minimal. In particular, the counterexample provided by [Luttmer and Mariotti \(2003\)](#), which is a continuous dynamic game with perfect information and Nature, does not have any subgame-perfect equilibrium. In their example, Nature is active in the third period, but the state transitions could have atoms. Thus, our condition of atomless transitions is violated.*

The next corollary follows from Proposition 3, which presents the existence result for continuous dynamic games with perfect information (and Nature).

Corollary 3. *If a continuous dynamic game with perfect information has atomless transitions, then it possesses a pure-strategy subgame-perfect equilibrium.*

Remark 6. *[Harris \(1985\)](#), [Hellwig and Leininger \(1987\)](#), [Börgers \(1989\)](#) and [Hellwig et al. \(1990\)](#) proved the existence of subgame-perfect equilibria in continuous dynamic games with perfect information. In particular, Nature is absent in all those papers. [Luttmer and Mariotti \(2003\)](#) provided an example of a five-stage continuous dynamic game with perfect information, in which Nature is present and no subgame-perfect equilibrium exists. The only known general existence result, to the best of our knowledge, for (finite or infinite horizon) continuous dynamic games with perfect information and Nature is the existence of subgame-perfect correlated equilibria via public randomization as in [Harris, Reny and Robson \(1995\)](#). Corollary 3 covers all those existence results as special cases.*

5 Appendix

In Subsection 5.1, we present Lemmas 1-15 as the mathematical preparations for proving Theorem 1. Since correspondences will be used extensively in the proofs, we collect, for the convenience of the reader, several known results on various properties of correspondences as Lemmas 1-7. The proof for the finite-horizon case of Theorem 1 will depend on Lemmas 1-7. The rest of the lemmas (Lemmas 8-15) will only be used for proving the infinite-horizon case.

We present in Subsection 5.2 a new existence result for discontinuous games with stochastic endogenous sharing rules. The proofs of Theorem 1 and Proposition 1 are given in Subsection 5.3. Finally, the proofs for the results in Section 4 are provided in Subsections 5.4 and 5.5.

5.1 Technical preparations

Let (S, \mathcal{S}) be a measurable space and X a topological space with its Borel σ -algebra $\mathcal{B}(X)$. A correspondence Ψ from S to X is a function from S to the space of all subsets of X . The upper inverse Ψ^u of a subset $A \subseteq X$ is

$$\Psi^u(A) = \{s \in S : \Psi(s) \subseteq A\}.$$

The lower inverse Ψ^l of a subset $A \subseteq X$ is

$$\Psi^l(A) = \{s \in S : \Psi(s) \cap A \neq \emptyset\}.$$

The correspondence Ψ is

1. weakly measurable, if $\Psi^l(O) \in \mathcal{S}$ for each open subset $O \subseteq X$;
2. measurable, if $\Psi^l(K) \in \mathcal{S}$ for each closed subset $K \subseteq X$.

The graph of Ψ is denoted by $\text{Gr}(\Psi) = \{(s, x) \in S \times X : s \in S, x \in \Psi(s)\}$. The correspondence Ψ is said to have a measurable graph if $\text{Gr}(\Psi) \in \mathcal{S} \otimes \mathcal{B}(X)$.

If S is a topological space, then Ψ is

1. upper hemicontinuous, if $\Psi^u(O)$ is open for each open subset $O \subseteq X$;
2. lower hemicontinuous, if $\Psi^l(O)$ is open for each open subset $O \subseteq X$.

The following lemma shows that the space of nonempty compact subsets of a Polish space is still Polish under the Hausdorff metric topology.

Lemma 1. *Suppose that X is a Polish space and \mathcal{K} is the set of all nonempty compact subsets of X endowed with the Hausdorff metric topology. Then \mathcal{K} is a Polish space.*

Proof. By Theorem 3.88 (2) of [Aliprantis and Border \(2006\)](#), \mathcal{K} is complete. In addition, Corollary 3.90 and Theorem 3.91 of [Aliprantis and Border \(2006\)](#) imply that \mathcal{K} is separable. Thus, \mathcal{K} is a Polish space. \square

The following two lemmas present some basic measurability and continuity properties for correspondences.

Lemma 2. *Let (S, \mathcal{S}) be a measurable space, X a Polish space endowed with the Borel σ -algebra $\mathcal{B}(X)$, and \mathcal{K} the space of nonempty compact subsets of X endowed with its Hausdorff metric topology. Suppose that $\Psi: S \rightarrow \mathcal{K}$ is a nonempty and closed valued correspondence.*

1. *If Ψ is weakly measurable, then it has a measurable graph.*
2. *If Ψ is compact valued, then the following statements are equivalent.*
 - (a) *The correspondence Ψ is weakly measurable.*
 - (b) *The correspondence Ψ is measurable.*
 - (c) *The function $f: S \rightarrow \mathcal{K}$, defined by $f(s) = \Psi(s)$, is Borel measurable.*
3. *Suppose that S is a topological space. If Ψ is compact valued, then the function $f: S \rightarrow \mathcal{K}$ defined by $f(s) = \Psi(s)$ is continuous if and only if the correspondence Ψ is continuous.*
4. *Suppose that $(S, \mathcal{S}, \lambda)$ is a complete probability space. Then Ψ is \mathcal{S} -measurable if and only if it has a measurable graph.*
5. *For a correspondence $\Psi: S \rightarrow \mathcal{K}$ between two Polish spaces, the following statements are equivalent.*
 - (a) *The correspondence Ψ is upper hemicontinuous at a point $s \in S$ and $\Psi(s)$ is compact.*
 - (b) *If a sequence (s_n, x_n) in the graph of Ψ satisfies $s_n \rightarrow s$, then the sequence $\{x_n\}$ has a limit in $\Psi(s)$.*
6. *For a correspondence $\Psi: S \rightarrow \mathcal{K}$ between two Polish spaces, the following statements are equivalent.*
 - (a) *The correspondence Ψ is lower hemicontinuous at a point $s \in S$.*
 - (b) *If $s_n \rightarrow s$, then for each $x \in \Psi(s)$, there exist a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ and elements $x_k \in \Psi(s_{n_k})$ for each k such that $x_k \rightarrow x$.*
7. *Given correspondences $F: X \rightarrow Y$ and $G: Y \rightarrow Z$, the composition F and G is defined by*

$$G(F(x)) = \cup_{y \in F(x)} G(y).$$

The composition of upper hemicontinuous correspondences is upper hemicontinuous. The composition of lower hemicontinuous correspondences is lower hemicontinuous.

Proof. Properties (1), (2), (3), (5), (6) and (7) are Theorems 18.6, 18.10, 17.15, 17.20, 17.21 and 17.23 of [Aliprantis and Border \(2006\)](#), respectively. Property (4) follows from Proposition 4 of [Hildenbrand \(1974, p.61\)](#). \square

- Lemma 3.**
1. *A correspondence Ψ from a measurable space (S, \mathcal{S}) into a topological space X is weakly measurable if and only if its closure correspondence $\overline{\Psi}$ is weakly measurable, where for each $s \in S$, $\overline{\Psi}(s) = \overline{\Psi(s)}$ and $\overline{\Psi(s)}$ is the closure of the set $\Psi(s)$ in X .*
 2. *For a sequence $\{\Psi_m\}$ of correspondences from a measurable space (S, \mathcal{S}) into a Polish space, the union correspondence $\Psi(s) = \cup_{m \geq 1} \Psi_m(s)$ is weakly measurable if each Ψ_m is weakly measurable. If each Ψ_m is weakly measurable and compact valued, then the intersection correspondence $\Phi(s) = \cap_{m \geq 1} \Psi_m(s)$ is weakly measurable.*
 3. *A weakly measurable, nonempty and closed valued correspondence from a measurable space into a Polish space admits a measurable selection.*
 4. *A correspondence with closed graph between compact metric spaces is measurable.*
 5. *A nonempty and compact valued correspondence Ψ from a measurable space (S, \mathcal{S}) into a Polish space is weakly measurable if and only if there exists a sequence $\{\psi_1, \psi_2, \dots\}$ of measurable selections of Ψ such that $\Psi(s) = \overline{\{\psi_1(s), \psi_2(s), \dots\}}$ for each $s \in S$.*
 6. *The image of a compact set under a compact valued upper hemicontinuous correspondence is compact.²² If the domain is compact, then the graph of a compact valued upper hemicontinuous correspondence is compact.*
 7. *The intersection of a correspondence with closed graph and an upper hemicontinuous compact valued correspondence is upper hemicontinuous.*
 8. *If the correspondence $\Psi: S \rightarrow \mathbb{R}^l$ is compact valued and upper hemicontinuous, then the convex hull of Ψ is also compact valued and upper hemicontinuous.*

Proof. Properties (1)-(7) are Lemmas 18.3 and 18.4, Theorems 18.13 and 18.20, Corollary 18.15, Lemma 17.8 and Theorem 17.25 in [Aliprantis and Border \(2006\)](#), respectively. Property 8 is Proposition 6 in [Hildenbrand \(1974, p.26\)](#). \square

²²Given a correspondence $F: X \rightarrow Y$ and a subset A of X , the image of A under F is defined to be the set $\cup_{x \in A} F(x)$.

Parts 1 and 2 of the following lemma are the standard Lusin's Theorem and Michael's continuous selection theorem, while the other two parts are about properties involving mixture of measurability and continuity of correspondences.

- Lemma 4.** 1. *Lusin's Theorem: Suppose that S is a Borel subset of a Polish space, λ is a Borel probability measure on S and \mathcal{S} is the completion of $\mathcal{B}(S)$ under λ . Let X be a Polish space. If f is an \mathcal{S} -measurable mapping from S to X , then for any $\epsilon > 0$, there exists a compact subset $S_1 \subseteq S$ with $\lambda(S \setminus S_1) < \epsilon$ such that the restriction of f to S_1 is continuous.*
2. *Michael's continuous selection theorem: Let S be a metrizable space, and X a complete metrizable closed subset of some locally convex space. Suppose that $F: S \rightarrow X$ is a lower hemicontinuous, nonempty, convex and closed valued correspondence. Then there exists a continuous mapping $f: S \rightarrow X$ such that $f(s) \in F(s)$ for all $s \in S$.*
3. *Let $(S, \mathcal{S}, \lambda)$ be a finite measure space, X a Polish space, and Y a locally convex linear topological space. Let $F: S \rightarrow X$ be a closed-valued correspondence such that $\text{Gr}(F) \in \mathcal{S} \otimes \mathcal{B}(X)$, and $f: \text{Gr}(F) \rightarrow Y$ a measurable function which is sectionally continuous on X . Then there exists a measurable function $f': S \times X \rightarrow Y$ such that (1) f' is sectionally continuous on X , (2) for λ -almost all $s \in S$, $f'(s, x) = f(s, x)$ for all $x \in F(s)$ and $f'(s, X) \subseteq \text{co}f(s, F(s))$.²³*
4. *Let (S, \mathcal{S}) be a measurable space, and X and Y Polish spaces. Let $\Psi: S \times X \rightarrow \mathcal{M}(Y)$ be an $\mathcal{S} \otimes \mathcal{B}(X)$ -measurable, nonempty, convex and compact valued correspondence which is sectionally continuous on X , where the compactness and continuity are with respect to the weak* topology on $\mathcal{M}(Y)$. Then there exists an $\mathcal{S} \otimes \mathcal{B}(X)$ -measurable selection ψ of Ψ that is sectionally continuous on X .*

Proof. Lusin's theorem is Theorem 7.1.13 in [Bogachev \(2007\)](#). Michael's continuous selection theorem can be found in [Michael \(1966\)](#) and the last paragraph of [Bogachev \(2007, p.228\)](#). Properties (3) is Theorem 2.7 in [Brown and Schreiber \(1989\)](#). Property (4) follows from Theorem 1 and the main lemma of [Kucia \(1998\)](#). \square

The following lemma presents the convexity, compactness and continuity properties on integration of correspondences.

²³For any set A in a linear topological space, $\text{co}A$ denotes the convex hull of A .

Lemma 5. *Let $(S, \mathcal{S}, \lambda)$ be an atomless probability space, X a Polish space, and F a correspondence from S to \mathbb{R}^l . Denote*

$$\int_S F(s) \lambda(ds) = \left\{ \int_S f(s) \lambda(ds) : f \text{ is an integrable selection of } F \text{ on } S \right\}.$$

1. *If F is measurable, nonempty and closed valued, and λ -integrably bounded by some integrable function $\psi: S \rightarrow \mathbb{R}_+$ in the sense that for λ -almost all $s \in S$, $\|y\| \leq \psi(s)$ for any $y \in F(s)$, then $\int_S F(s) \lambda(ds)$ is nonempty, convex and compact, and*

$$\int_S F(s) \lambda(ds) = \int_S \text{co}F(s) \lambda(ds).$$

2. *If G is a measurable, nonempty and closed valued correspondence from $S \times X \rightarrow \mathbb{R}^l$ such that (1) $G(s, \cdot)$ is upper (resp. lower) hemicontinuous on X for all $s \in S$, and (2) G is λ -integrably bounded by some integrable function $\psi: S \rightarrow \mathbb{R}_+$ in the sense that for λ -almost all $s \in S$, $\|y\| \leq \psi(s)$ for any $x \in X$ and $y \in G(s, x)$, then $\int_S G(s, x) \lambda(ds)$ is upper (resp. lower) hemicontinuous on X .*

Proof. See Theorems 2, 3 and 4, Propositions 7 and 8, and Problem 6 in Section D.II.4 of [Hildenbrand \(1974\)](#). \square

The following result proves a measurable version of Lyapunov's theorem, which is taken from [Mertens \(2003\)](#). Let (S, \mathcal{S}) and (X, \mathcal{X}) be measurable spaces. A transition probability from S to X is a mapping f from S to the space $\mathcal{M}(X)$ of probability measures on (X, \mathcal{X}) such that $f(B|\cdot) : s \rightarrow f(B|s)$ is \mathcal{S} -measurable for each $B \in \mathcal{X}$.

Lemma 6. *Let $f(\cdot|s)$ be a transition probability from a measurable space (S, \mathcal{S}) to another measurable space (X, \mathcal{X}) (\mathcal{X} is separable).²⁴ Let Q be a measurable, nonempty and compact valued correspondence from $S \times X$ to \mathbb{R}^l , which is f -integrable in the sense that for any measurable selection q of Q , $q(\cdot, s)$ is $f(\cdot|s)$ -absolutely integrable for any $s \in S$. Let $\int Q df$ be the correspondence from S to subsets of \mathbb{R}^l defined by*

$$M(s) = \left(\int Q df \right) (s) = \left\{ \int_X q(s, x) f(dx|s) : q \text{ is a measurable selection of } Q \right\}.$$

Denote the graph of M by J . Let \mathcal{J} be the restriction of the product σ -algebra $\mathcal{S} \otimes \mathcal{B}(\mathbb{R}^l)$ to J .

Then

²⁴A σ -algebra is said to be separable if it is generated by a countable collection of sets.

1. M is a measurable, nonempty and compact valued correspondence;
2. there exists a measurable, \mathbb{R}^l -valued function g on $(X \times J, \mathcal{X} \otimes \mathcal{J})$ such that $g(x, e, s) \in Q(x, s)$ and $e = \int_X g(x, e, s) f(dx|s)$.

The following result is Lemma 6 of [Reny and Robson \(2002\)](#).

Lemma 7. *Suppose that H and X are compact metric spaces. Let $P: H \times X \rightarrow \mathbb{R}^n$ be a nonempty valued and upper hemicontinuous correspondence, and the mappings $f: H \rightarrow \mathcal{M}(X)$ and $\mu: H \rightarrow \Delta(X)$ be measurable. In addition, suppose that $\mu(\cdot|h) = p(h, \cdot) \circ f(\cdot|h)$ such that $p(h, \cdot)$ is a measurable selection of $P(h, \cdot)$. Then there exists a jointly Borel measurable selection g of P such that $\mu(\cdot|h) = g(h, \cdot) \circ f(\cdot|h)$; that is, $g(h, x) = p(h, x)$ for $f(\cdot|h)$ -almost all x .*

Suppose that (S_1, \mathcal{S}_1) is a measurable space, S_2 is a Polish space endowed with the Borel σ -algebra, and $S = S_1 \times S_2$ which is endowed with the product σ -algebra \mathcal{S} . Let D be an \mathcal{S} -measurable subset of S such that $D(s_1)$ is compact for any $s_1 \in S_1$. The σ -algebra \mathcal{D} is the restriction of \mathcal{S} on D . Let X be a Polish space, and A a \mathcal{D} -measurable, nonempty and closed valued correspondence from D to X which is sectionally continuous on S_2 . The following lemma considers the property of upper hemicontinuity for the correspondence M as defined in Lemma 6.

Lemma 8. *Let $f(\cdot|s)$ be a transition probability from (D, \mathcal{D}) to $\mathcal{M}(X)$ such that $\int f(A(s)|s) = 1$ for any $s \in D$, which is sectionally continuous on S_2 . Let G be a bounded, measurable, nonempty, convex and compact valued correspondence from $\text{Gr}(A)$ to \mathbb{R}^l , which is sectionally upper hemicontinuous on $S_2 \times X$. Let $\int G df$ be the correspondence from D to subsets of \mathbb{R}^l defined by*

$$M(s) = \left(\int G df \right) (s) = \left\{ \int_X g(s, x) f(dx|s) : g \text{ is a measurable selection of } G \right\}.$$

Then M is \mathcal{S} -measurable, nonempty and compact valued, and sectionally upper hemicontinuous on S_2 .

Proof. Define a correspondence $\tilde{G}: S \times X \rightarrow \mathbb{R}^l$ as

$$\tilde{G} = \begin{cases} G(s, x), & \text{if } (s, x) \in \text{Gr}(A); \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then $M(s) = \left(\int \tilde{G} df \right) (s) = \left(\int G df \right) (s)$. The measurability, nonemptiness and compactness follows from Lemma 6. Given $s_1 \in S_1$ such that (1) $D(s_1) \neq \emptyset$, (2) $f(s_1, \cdot)$ and $G(s_1, \cdot, \cdot)$ is upper hemicontinuous. The upper hemicontinuity of $M(s_1, \cdot)$ follows from Lemma 2 in [Simon and Zame \(1990\)](#) and Lemma 4 in [Reny and Robson \(2002\)](#). \square

From now onwards, whenever we work with mappings taking values in the space $\mathcal{M}(X)$ of Borel probability measures on some separable metric space X , the relevant continuity or convergence is assumed to be in terms of the topology of weak convergence of measures on $\mathcal{M}(X)$ unless otherwise noted.

In the following lemma, we state some properties for transition correspondences.

Lemma 9. *Suppose that Y and Z are Polish spaces. Let G be a measurable, nonempty, convex and compact valued correspondence from Y to $\mathcal{M}(Z)$. Define a correspondence G' from $\mathcal{M}(Y)$ to $\mathcal{M}(Z)$ as*

$$G'(\nu) = \left\{ \int_Y g(y)\nu(dy) : g \text{ is a Borel measurable selection of } G \right\}.$$
²⁵

1. *The correspondence G' is measurable, nonempty, convex and compact valued.*
2. *The correspondence G is upper hemicontinuous if and only if G' is upper hemicontinuous. In addition, if G is continuous, then G' is continuous.*

Proof. (1) is Lemma 19.29 of [Aliprantis and Border \(2006\)](#). By Theorem 19.30 therein, G is upper hemicontinuous if and only if G' is upper hemicontinuous. We need to show that G' is lower hemicontinuous if G is lower hemicontinuous.

Let Z be endowed with a totally bounded metric, and $U(Z)$ the space of bounded, real-valued and uniformly continuous functions on Z endowed with the supremum norm, which is obviously separable. Pick a countable set $\{f_m\}_{m \geq 1} \subseteq U(Z)$ such that $\{f_m\}$ is dense in the unit ball of $U(Z)$. It follows from Theorem 15.2 of [Aliprantis and Border \(2006\)](#) that the weak* topology of $\mathcal{M}(Z)$ is metrizable by the metric d_z , where

$$d_z(\mu_1, \mu_2) = \sum_{m=1}^{\infty} \frac{1}{2^m} \left| \int_Z f_m(z)\mu_1(dz) - \int_Z f_m(z)\mu_2(dz) \right|$$

for each pair of $\mu_1, \mu_2 \in \mathcal{M}(Z)$.

Suppose that $\{\nu_j\}_{j \geq 0}$ is a sequence in $\mathcal{M}(Y)$ such that $\nu_j \rightarrow \nu_0$ as $j \rightarrow \infty$. Pick an arbitrary point $\mu_0 \in G'(\nu_0)$. By the definition of G' , there exists a Borel measurable selection g of G such that $\mu_0 = \int_Y g(y)\nu_0(dy)$.

For each $k \geq 1$, by Lemma 4 (Lusin's theorem), there exists a compact subset $D_k \subseteq Y$ such that g is continuous on D_k and $\nu_0(Y \setminus D_k) < \frac{1}{3k}$. Define a

²⁵The integral $\int_Y g(y)\nu(dy)$ defines a Borel probability measure τ on Z such that for any Borel set C in Z , $\tau(C) = \int_Y g(y)(C)\nu(dy)$. The measure τ is also equal to the Gelfand integral of g with respect to the measure ν on Y , when $\mathcal{M}(Z)$ is viewed as a set in the dual space of the space of bounded continuous functions on Z ; see Definition 11.49 of [Aliprantis and Border \(2006\)](#).

correspondence $G_k: Y \rightarrow \mathcal{M}(X)$ as follows:

$$G_k(y) = \begin{cases} \{g(y)\}, & y \in D_k; \\ G(y), & y \in Y \setminus D_k. \end{cases}$$

Then G_k is nonempty, convex and compact valued, and lower hemicontinuous. By Theorem 3.22 in Aliprantis and Border (2006), Y is paracompact. Then by Lemma 4 (Michael's selection theorem), it has a continuous selection g_k .

For each k , since $\nu_j \rightarrow \nu_0$ and g_k is continuous, $\int_Y g_k(y)\nu_j(dy) \rightarrow \int_Y g_k(y)\nu_0(dy)$ in the sense that for any $m \geq 1$,

$$\int_Y \int_Z f_m(z)g_k(dz|y)\nu_j(dy) \rightarrow \int_Y \int_Z f_m(z)g_k(dz|y)\nu_0(dy).$$

Thus, there exists a point ν_{j_k} such that $\{j_k\}$ is an increasing sequence and

$$d_z \left(\int_Y g_k(y)\nu_{j_k}(dy), \int_Y g_k(y)\nu_0(dy) \right) < \frac{1}{3k}.$$

In addition, since g_k coincides with g on D_k and $\nu_0(Y \setminus D_k) < \frac{1}{3k}$,

$$d_z \left(\int_Y g_k(y)\nu_0(dy), \int_Y g(y)\nu_0(dy) \right) < \frac{2}{3k}.$$

Thus,

$$d_z \left(\int_Y g_k(y)\nu_{j_k}(dy), \int_Y g(y)\nu_0(dy) \right) < \frac{1}{k}.$$

Let $\mu_{j_k} = \int_Y g_k(y)\nu_{j_k}(dy)$ for each k . Then $\mu_{j_k} \in G'(\nu_{j_k})$ and $\mu_{j_k} \rightarrow \mu_0$ as $k \rightarrow \infty$. By Lemma 2, G' is lower hemicontinuous. \square

The next lemma presents some properties for the composition of two transition correspondences in terms of the product of transition probabilities.

Lemma 10. *Let X, Y and Z be Polish spaces, and G a measurable, nonempty and compact valued correspondence from X to $\mathcal{M}(Y)$. Suppose that F is a measurable, nonempty, convex and compact valued correspondence from $X \times Y$ to $\mathcal{M}(Z)$. Define a correspondence Π from X to $\mathcal{M}(Y \times Z)$ as follows:*

$$\Pi(x) = \{g(x) \diamond f(x) : g \text{ is a Borel measurable selection of } G, \\ f \text{ is a Borel measurable selection of } F\}.$$

1. *If F is sectionally continuous on Y , then Π is a measurable, nonempty and compact valued correspondence.*

2. If there exists a function g from X to $\mathcal{M}(Y)$ such that $G(x) = \{g(x)\}$ for any $x \in X$, then Π is a measurable, nonempty and compact valued correspondence.
3. If both G and F are continuous correspondences, then Π is a nonempty and compact valued, and continuous correspondence.²⁶
4. If $G(x) \equiv \{\lambda\}$ for some fixed Borel probability measure $\lambda \in \mathcal{M}(Y)$ and F is sectionally continuous on X , then Π is a continuous, nonempty and compact valued correspondence.

Proof. (1) Define three correspondences $\tilde{F}: X \times Y \rightarrow \mathcal{M}(Y \times Z)$, $\hat{F}: \mathcal{M}(X \times Y) \rightarrow \mathcal{M}(Y \times Z)$ and $\check{F}: X \times \mathcal{M}(Y) \rightarrow \mathcal{M}(Y \times Z)$ as follows:

$$\begin{aligned}\tilde{F}(x, y) &= \{\delta_y \otimes \mu: \mu \in F(x, y)\}, \\ \hat{F}(\tau) &= \left\{ \int_{X \times Y} f(x, y) \tau(d(x, y)): f \text{ is a Borel measurable selection of } \tilde{F} \right\}, \\ \check{F}(x, \mu) &= \hat{F}(\delta_x \otimes \mu).\end{aligned}$$

Since F is measurable, nonempty, convex and compact valued, \tilde{F} is measurable, nonempty, convex and compact valued. By Lemma 9, the correspondence \hat{F} is measurable, nonempty, convex and compact valued, and $\check{F}(x, \cdot)$ is continuous on $\mathcal{M}(Y)$ for any $x \in X$.

Since G is measurable and compact valued, there exists a sequence of Borel measurable selections $\{g_k\}_{k \geq 1}$ of G such that $G(x) = \overline{\{g_1(x), g_2(x), \dots\}}$ for any $x \in X$ by Lemma 3 (5). For each $k \geq 1$, define a correspondence Π^k from X to $\mathcal{M}(Y \times Z)$ by letting $\Pi^k(x) = \check{F}(x, g_k(x)) = \hat{F}(x \otimes g_k(x))$. Then Π^k is measurable, nonempty, convex and compact valued.

Fix any $x \in X$. It is clear that $\Pi(x) = \check{F}(x, G(x))$ is a nonempty valued. Since $G(x)$ is compact, and $\check{F}(x, \cdot)$ is compact valued and continuous, $\Pi(x)$ is compact by Lemma 3. Thus, the closure $\overline{\bigcup_{k \geq 1} \Pi^k(x)}$ of $\bigcup_{k \geq 1} \Pi^k(x)$ is a subset of $\Pi(x)$.

Fix any $x \in X$ and $\tau \in \Pi(x)$. There exists a point $\nu \in G(x)$ such that $\tau \in \check{F}(x, \nu)$. Since $\{g_k(x)\}_{k \geq 1}$ is dense in $G(x)$, it has a subsequence $\{g_{k_m}(x)\}$ such that $g_{k_m}(x) \rightarrow \nu$. As $\check{F}(x, \cdot)$ is continuous, $\check{F}(x, g_{k_m}(x)) \rightarrow \check{F}(x, \nu)$. That is,

$$\tau \in \overline{\bigcup_{k \geq 1} \check{F}(x, g_k(x))} = \overline{\bigcup_{k \geq 1} \Pi^k(x)}.$$

Therefore, $\overline{\bigcup_{k \geq 1} \Pi^k(x)} = \Pi(x)$ for any $x \in X$. Lemma 3 (1) and (2) imply that Π is measurable.

²⁶In Lemma 29 of [Harris, Reny and Robson \(1995\)](#), they showed that Π is upper hemicontinuous if both G and F are upper hemicontinuous.

(2) As in (1), the correspondence \hat{F} is measurable, nonempty, convex and compact valued. If $G = \{g\}$ for some measurable function g , then $\Pi(x) = \hat{F}(\delta_x \otimes g(x))$, which is measurable, nonempty and compact valued.

(3) We continue to work with the two correspondences $\tilde{F}: X \times Y \rightarrow \mathcal{M}(Y \times Z)$ and $\hat{F}: \mathcal{M}(X \times Y) \rightarrow \mathcal{M}(Y \times Z)$ as in Part (1). By the condition on F , it is obvious that the correspondence \tilde{F} is continuous, nonempty, convex and compact valued. Lemma 9 implies the properties for the correspondence \hat{F} . Define a correspondence $\hat{G}: X \rightarrow \mathcal{M}(X \times Y)$ as $\hat{G}(x) = \delta_x \otimes G(x)$. Since \hat{G} and \hat{F} are both nonempty valued, $\Pi(x) = \hat{F}(\hat{G}(x))$ is nonempty. As \hat{G} is compact valued and \hat{F} is continuous, Π is compact valued by Lemma 3. As \hat{G} and \hat{F} are both continuous, Π is continuous by Lemma 2 (7).

(4) Let $Y' = Y$ and define a correspondence $\check{F}: X \times Y \rightarrow \mathcal{M}(Y' \times Z)$ as follows:

$$\check{F}(x, y) = \{\delta_y \otimes \mu : \mu \in F(x, y)\}.$$

Then \check{F} is also measurable, nonempty, convex and compact valued, and sectionally upper hemicontinuous on X .

Let d be a totally bounded metric on $Y' \times Z$, and $U(Y' \times Z)$ the space of bounded, real-valued and uniformly continuous functions on $Y' \times Z$ endowed with the supremum norm, which is obviously separable. It follows from Theorem 15.2 of Aliprantis and Border (2006) that the space of Borel probability measures on $Y' \times Z$ with the topology of weak convergence of measures can be viewed as a subspace of the dual space of $U(Y' \times Z)$ with the weak* topology. By Corollary 18.37 of Aliprantis and Border (2006), $\Pi(x) = \int_Y \check{F}(x, y) \lambda(dy)$ is nonempty, convex and compact for any $x \in X$.²⁷

Now we shall show the upper hemicontinuity. If $x_n \rightarrow x_0$ and $\mu_n \in \Pi(x_n)$, we need to prove that there exists some $\mu_0 \in \Pi(x_0)$ such that a subsequence of $\{\mu_n\}$ weakly converges to μ_0 . Suppose that for $n \geq 1$, f_n is a Borel measurable selection of $F(x_n, \cdot)$ such that $\mu_n = \lambda \diamond f_n$.

Fix any $y \in Y$, let $J(y) = \overline{\text{co}}\{f_n(y) \otimes \delta_{x_n}\}_{n \geq 1}$, which is the closure of the convex hull of $\{f_n(y) \otimes \delta_{x_n}\}_{n \geq 1}$. It is obvious that $J(y)$ is nonempty and convex. It is also clear that $J(y)$ is the closure of the countable set

$$\left\{ \sum_{i=1}^n \alpha_i f_i(y) \otimes \delta_{x_i} : n \geq 1, \alpha_i \in \mathbb{Q}_+, i = 1, \dots, n, \sum_{i=1}^n \alpha_i = 1 \right\},$$

²⁷Note that the integral $\int_Y \check{F}(x, y) \lambda(dy)$ can be viewed as the Gelfand integral of the correspondence in the dual space of $U(Y' \times Z)$; see Definition 18.36 of Aliprantis and Border (2006).

where \mathbb{Q}_+ is the set of non-negative rational numbers. Let $F'(x) = \{\mu \otimes \delta_x : \mu \in F(x, y)\}$ for any $x \in X$. Then, F' is continuous on X . Since $\{x_n : n \geq 0\}$ is a compact set, Lemma 3 (6) implies that $\bigcup_{n \geq 0} F'(x_n)$ is compact. Hence, $\{f_n(y) \otimes \delta_{x_n}\}_{n \geq 1}$ is relatively compact. By Theorem 5.22 of Aliprantis and Border (2006), $\{f_n(y) \otimes \delta_{x_n}\}_{n \geq 1}$ is tight. That is, for any positive real number ϵ , there is a compact set K_ϵ in $Z \times X$ such that for any $n \geq 1$, $f_n(y) \otimes \delta_{x_n}(K_\epsilon) > 1 - \epsilon$. Thus, $\sum_{i=1}^n \alpha_i f_i(y) \otimes \delta_{x_i}(K_\epsilon) > 1 - \epsilon$ for any $n \geq 1$, and for any $\alpha_i \in \mathbb{Q}_+$, $i = 1, \dots, n$ with $\sum_{i=1}^n \alpha_i = 1$. Hence, $J(y)$ is compact by Theorem 5.22 of Aliprantis and Border (2006) again.

For any $n \geq 1$, and for any $\alpha_i \in \mathbb{Q}_+$, $i = 1, \dots, n$ with $\sum_{i=1}^n \alpha_i = 1$, it is clear that $\sum_{i=1}^n \alpha_i f_i(y) \otimes \delta_{x_i}$ is measurable in $y \in Y$. Lemma 3 (5) implies that J is also a measurable correspondence from Y to $\mathcal{M}(Z \times X)$. By the argument in the second paragraph of the proof of Part (4), the set

$$\Lambda = \{\lambda \diamond \zeta : \zeta \text{ is a Borel measurable selection of } J\}$$

is compact.

Since $\lambda \diamond (f_n \otimes \delta_{x_n}) \in \Lambda$ for each n , there exists some Borel measurable selection ζ of J such that a subsequence of $\{\lambda \diamond (f_n \otimes \delta_{x_n})\}$, say itself, weakly converges to $\lambda \diamond \zeta \in \Lambda$. Let $\zeta_X(y)$ be the marginal probability of $\zeta(y)$ on X for each y . Since x_n converges to x_0 , $\zeta_X(y) = \delta_{x_0}$ for λ -almost all $y \in Y$. As a result, there exists a Borel measurable function f_0 such that $\zeta = f_0 \otimes \delta_{x_0}$, where $f_0(y) \in \overline{\text{co}}\text{Ls}_n\{f_n(y)\}$ for λ -almost all $y \in Y$. Since F is convex and compact valued, and upper hemicontinuous on X , f_0 is a measurable selection of $F(x_0, \cdot)$. Let $\mu_0 = \lambda \diamond f_0$. Then μ_n weakly converges to μ_0 , which implies that Π is upper hemicontinuous.

Next we shall show the lower hemicontinuity of Π . Suppose that $x_n \rightarrow x_0$ and $\mu_0 \in \Pi(x_0)$. Then there exists a Borel measurable selection f_0 of $F(x_0, \cdot)$ such that $\mu_0 = \lambda \diamond f_0$. Since F is sectionally lower hemicontinuous on X and compact valued, for each $n \geq 1$, there exists a measurable selection f_n for $F(x_n, \cdot)$ such that $f_n(y)$ weakly converges to $f_0(y)$ for each $y \in Y$.²⁸ Denote $\mu_n = \lambda \diamond f_n$. For any bounded continuous function ψ on $Y \times Z$, $\int_Z \psi(y, z) f_n(dz|y)$ converges to $\int_Z \psi(y, z) f_0(dz|y)$ for any $y \in Y$. Thus, we have

$$\begin{aligned} \int_{Y \times Z} \psi(y, z) \mu_n(d(y, z)) &= \int_Y \int_Z \psi(y, z) f_n(dz|y) \lambda(dy) \\ &\rightarrow \int_Y \int_Z \psi(y, z) f_0(dz|y) \lambda(dy) \end{aligned}$$

by the Dominated Convergence Theorem. Therefore, Π is lower hemicontinuous.

²⁸See Proposition 4.2 in Sun (1997). Notice that the atomless Loeb probability measurable space assumption is not needed for the result of lower hemicontinuity as in Theorem 10 therein.

The proof is thus complete. \square

The following result presents a variant of Lemma 6 in terms of transition correspondences.

Lemma 11. *Let X and Y be Polish spaces, and Z a compact subset of \mathbb{R}_+^l . Let G be a measurable, nonempty and compact valued correspondence from X to $\mathcal{M}(Y)$. Suppose that F is a measurable, nonempty, convex and compact valued correspondence from $X \times Y$ to Z . Define a correspondence Π from X to Z as follows:*

$$\Pi(x) = \left\{ \int_Y f(x, y)g(dy|x) : g \text{ is a Borel measurable selection of } G, \right. \\ \left. f \text{ is a Borel measurable selection of } F \right\}.$$

If F is sectionally continuous on Y , then

1. the correspondence $\tilde{F}: X \times \mathcal{M}(Y) \rightarrow Z$ as $\tilde{F}(x, \nu) = \int_Y F(x, y)\nu(dy)$ is sectionally continuous on $\mathcal{M}(Y)$; and
2. Π is a measurable, nonempty and compact valued correspondence.
3. If F and G are both continuous, then Π is continuous.

Proof. (1) For any fixed $x \in X$, the upper hemicontinuity of $\tilde{F}(x, \cdot)$ follows from Lemma 8.

Next, we shall show the lower hemicontinuity. Fix any $x \in X$. Suppose that $\{\nu_j\}_{j \geq 0}$ is a sequence in $\mathcal{M}(Y)$ such that $\nu_j \rightarrow \nu_0$ as $j \rightarrow \infty$. Pick an arbitrary point $z_0 \in \tilde{F}(x, \nu_0)$. Then there exists a Borel measurable selection f of $F(x, \cdot)$ such that $z_0 = \int_Y f(y)\nu_0(dy)$.

By Lemma 4 (Lusin's theorem), for each $k \geq 1$, there exists a compact subset $D_k \subseteq Y$ such that f is continuous on D_k and $\nu_0(Y \setminus D_k) < \frac{1}{3kM}$, where $M > 0$ is the bound of Z . Define a correspondence $F_k: Y \rightarrow Z$ as follows:

$$F_k(y) = \begin{cases} \{f(y)\}, & y \in D_k; \\ F(x, y), & y \in Y \setminus D_k. \end{cases}$$

Then F_k is nonempty, convex and compact valued, and lower hemicontinuous. By Theorem 3.22 in Aliprantis and Border (2006), Y is paracompact. Then by Lemma 4 (Michael's selection theorem), F_k has a continuous selection f_k .

For each k , since $\nu_j \rightarrow \nu_0$ and f_k is bounded and continuous, $\int_Y f_k(y)\nu_j(dy) \rightarrow \int_Y f_k(y)\nu_0(dy)$ as $j \rightarrow \infty$. Thus, there exists a subsequence $\{\nu_{j_k}\}$ such that $\{j_k\}$

is an increasing sequence, and for each $k \geq 1$,

$$\left\| \int_Y f_k(y) \nu_{j_k}(\mathrm{d}y) - \int_Y f_k(y) \nu_0(\mathrm{d}y) \right\| < \frac{1}{3k},$$

where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^l .

Since f_k coincides with f on D_k , $\nu_0(Y \setminus D_k) < \frac{1}{3kM}$, and Z is bounded by M ,

$$\left\| \int_Y f_k(y) \nu_0(\mathrm{d}y) - \int_Y f(y) \nu_0(\mathrm{d}y) \right\| < \frac{2}{3k}.$$

Thus,

$$\left\| \int_Y f_k(y) \nu_{j_k}(\mathrm{d}y) - \int_Y f(y) \nu_0(\mathrm{d}y) \right\| < \frac{1}{k}.$$

Let $z_{j_k} = \int_Y f_k(y) \nu_{j_k}(\mathrm{d}y)$ for each k . Then $z_{j_k} \in \tilde{F}(x, \nu_{j_k})$ and $z_{j_k} \rightarrow z_0$ as $k \rightarrow \infty$. By Lemma 2, $\tilde{F}(x, \cdot)$ is lower hemicontinuous.

(2) Since G is measurable and compact valued, there exists a sequence of Borel measurable selections $\{g_k\}_{k \geq 1}$ of G such that $G(x) = \overline{\{g_1(x), g_2(x), \dots\}}$ for any $x \in X$ by Lemma 3 (5). For each $k \geq 1$, define a correspondence Π^k from X to Z by letting $\Pi^k(x) = \tilde{F}(x, g_k(x)) = \int_Y F(x, y) g_k(\mathrm{d}y|x)$. Since F is convex valued, so is Π^k . By Lemma 6, Π^k is also measurable, nonempty and compact valued.

Fix any $x \in X$. It is clear that $\Pi(x) = \tilde{F}(x, G(x))$ is nonempty valued. Since $G(x)$ is compact, and $\tilde{F}(x, \cdot)$ is compact valued and continuous, $\Pi(x)$ is compact by Lemma 3. Thus, $\overline{\bigcup_{k \geq 1} \Pi^k(x)} \subseteq \Pi(x)$.

Fix any $x \in X$ and $z \in \Pi(x)$. There exists a point $\nu \in G(x)$ such that $z \in \tilde{F}(x, \nu)$. Since $\{g_k(x)\}_{k \geq 1}$ is dense in $G(x)$, it has a subsequence $\{g_{k_m}(x)\}$ such that $g_{k_m}(x) \rightarrow \nu$. As $\tilde{F}(x, \cdot)$ is continuous, $\tilde{F}(x, g_{k_m}(x)) \rightarrow \tilde{F}(x, \nu)$. That is,

$$z \in \overline{\bigcup_{k \geq 1} \tilde{F}(x, g_k(x))} = \overline{\bigcup_{k \geq 1} \Pi^k(x)}.$$

Therefore, $\overline{\bigcup_{k \geq 1} \Pi^k(x)} = \Pi(x)$ for any $x \in X$. Lemma 3 (1) and (2) imply that Π is measurable.

(3) Define a correspondence $\hat{F}: \mathcal{M}(X \times Y) \rightarrow Z$ as follows:

$$\hat{F}(\tau) = \left\{ \int_{X \times Y} f(x, y) \tau(\mathrm{d}(x, y)) : f \text{ is a Borel measurable selection of } F \right\}.$$

By (1), \hat{F} is continuous. Define a correspondence $\hat{G}: X \rightarrow \mathcal{M}(X \times Y)$ as $\hat{G}(x) = \{\delta_x \otimes \nu : \nu \in G(x)\}$. Since \hat{G} and \hat{F} are both nonempty valued, $\Pi(x) = \hat{F}(\hat{G}(x))$ is

nonempty. As \hat{G} is compact valued and \hat{F} is continuous, Π is compact valued by Lemma 3. As \hat{G} and \hat{F} are both continuous, Π is continuous by Lemma 2 (7). \square

The following lemma shows that a measurable and sectionally continuous correspondence on a product space is approximately continuous on the product space.

Lemma 12. *Let S , X and Y be Polish spaces endowed with the Borel σ -algebras, and λ a Borel probability measure on S . Denote \mathcal{S} as the completion of the Borel σ -algebra $\mathcal{B}(S)$ of S under the probability measure λ . Suppose that D is a $\mathcal{B}(S) \otimes \mathcal{B}(Y)$ -measurable subset of $S \times Y$, where $D(s)$ is nonempty and compact for all $s \in S$. Let A be a nonempty and compact valued correspondence from D to X , which is sectionally continuous on Y and has a $\mathcal{B}(S \times Y \times X)$ -measurable graph. Then*

- (i) $\tilde{A}(s) = \text{Gr}(A(s, \cdot))$ is an \mathcal{S} -measurable mapping from S to the set of nonempty and compact subsets $\mathcal{K}_{Y \times X}$ of $Y \times X$;
- (ii) there exist countably many disjoint compact subsets $\{S_m\}_{m \geq 1}$ of S such that (1) $\lambda(\cup_{m \geq 1} S_m) = 1$, and (2) for each $m \geq 1$, $D_m = D \cap (S_m \times Y)$ is compact, and A is nonempty and compact valued, and continuous on each D_m .

Proof. (i) $A(s, \cdot)$ is continuous and $D(s)$ is compact, $\text{Gr}(A(s, \cdot)) \subseteq Y \times X$ is compact by Lemma 3. Thus, \tilde{A} is nonempty and compact valued. Since A has a measurable graph, \tilde{A} is an \mathcal{S} -measurable mapping from S to the set of nonempty and compact subsets $\mathcal{K}_{Y \times X}$ of $Y \times X$ by Lemma 2 (4).

(ii) Define a correspondence \tilde{D} from S to Y such that $\tilde{D}(s) = \{y \in Y : (s, y) \in D\}$. Then \tilde{D} is nonempty and compact valued. As in (i), \tilde{D} is \mathcal{S} -measurable. By Lemma 4 (Lusin's Theorem), there exists a compact subset $S_1 \subseteq S$ such that $\lambda(S \setminus S_1) < \frac{1}{2}$, \tilde{D} and \tilde{A} are continuous functions on S_1 . By Lemma 2 (3), \tilde{D} and \tilde{A} are continuous correspondences on S_1 . Let $D_1 = \{(s, y) \in D : s \in S_1, y \in \tilde{D}(s)\}$. Since S_1 is compact and \tilde{D} is continuous, D_1 is compact (see Lemma 3 (6)).

Following the same procedure, for any $m \geq 1$, there exists a compact subset $S_m \subseteq S$ such that (1) $S_m \cap (\cup_{1 \leq k \leq m-1} S_k) = \emptyset$ and $D_m = D \cap (S_m \times Y)$ is compact, (2) $\lambda(S_m) > 0$ and $\lambda(S \setminus (\cup_{1 \leq k \leq m} S_k)) < \frac{1}{2m}$, and (3) A is nonempty and compact valued, and continuous on D_m . This completes the proof. \square

The lemma below states an equivalence property for the weak convergence of Borel probability measures obtained from the product of transition probabilities.

Lemma 13. *Let S and X be Polish spaces, and λ a Borel probability measure on S . Suppose that $\{S_k\}_{k \geq 1}$ is a sequence of disjoint compact subsets of S such that $\lambda(\cup_{k \geq 1} S_k) = 1$. For each k , define a probability measure on S_k as $\lambda_k(D) = \frac{\lambda(D)}{\lambda(S_k)}$*

for any measurable subset $D \subseteq S_k$. Let $\{\nu_m\}_{m \geq 0}$ be a sequence of transition probabilities from S to $\mathcal{M}(X)$, and $\tau_m = \lambda \diamond \nu_m$ for any $m \geq 0$. Then τ_m weakly converges to τ_0 if and only if $\lambda_k \diamond \nu_m$ weakly converges to $\lambda_k \diamond \nu_0$ for each $k \geq 1$.

Proof. First, we assume that τ_m weakly converges to τ_0 . For any closed subset $E \subseteq S_k \times X$, we have $\limsup_{m \rightarrow \infty} \tau_m(E) \leq \tau_0(E)$. That is, $\limsup_{m \rightarrow \infty} \lambda \diamond \nu_m(E) \leq \lambda \diamond \nu_0(E)$. For any k , $\frac{1}{\lambda(S_k)} \limsup_{m \rightarrow \infty} \lambda \diamond \nu_m(E) \leq \frac{1}{\lambda(S_k)} \lambda \diamond \nu_0(E)$, which implies that $\limsup_{m \rightarrow \infty} \lambda_k \diamond \nu_m(E) \leq \lambda_k \diamond \nu_0(E)$. Thus, $\lambda_k \diamond \nu_m$ weakly converges to $\lambda_k \diamond \nu_0$ for each $k \geq 1$.

Second, we consider the case that $\lambda_k \diamond \nu_m$ weakly converges to $\lambda_k \diamond \nu_0$ for each $k \geq 1$. For any closed subset $E \subseteq S \times X$, let $E_k = E \cap (S_k \times X)$ for each $k \geq 1$. Then $\{E_k\}$ are disjoint closed subsets and $\limsup_{m \rightarrow \infty} \lambda_k \diamond \nu_m(E_k) \leq \lambda_k \diamond \nu_0(E_k)$. Since $\lambda_k \diamond \nu_m(E') = \frac{1}{\lambda(S_k)} \lambda \diamond \nu_m(E')$ for any k, m and measurable subset $E' \subseteq S_k \times X$, we have that $\limsup_{m \rightarrow \infty} \lambda \diamond \nu_m(E_k) \leq \lambda \diamond \nu_0(E_k)$. Thus,

$$\sum_{k \geq 1} \limsup_{m \rightarrow \infty} \lambda \diamond \nu_m(E_k) \leq \sum_{k \geq 1} \lambda \diamond \nu_0(E_k) = \lambda \diamond \nu_0(E).$$

Since the limit superior is subadditive, we have

$$\sum_{k \geq 1} \limsup_{m \rightarrow \infty} \lambda \diamond \nu_m(E_k) \geq \limsup_{m \rightarrow \infty} \sum_{k \geq 1} \lambda \diamond \nu_m(E_k) = \limsup_{m \rightarrow \infty} \lambda \diamond \nu_m(E).$$

Therefore, $\limsup_{m \rightarrow \infty} \lambda \diamond \nu_m(E) \leq \lambda \diamond \nu_0(E)$, which implies that τ_m weakly converges to τ_0 . \square

The following is a key lemma that allows one to drop the continuity condition on the state variables through a reference measure in Theorem 1.

Lemma 14. *Suppose that X, Y and S are Polish spaces and Z is a compact metric space. Let λ be a Borel probability measure on S , and A a nonempty and compact valued correspondence from $Z \times S$ to X which is sectionally upper hemicontinuous on Z and has a $\mathcal{B}(Z \times S \times X)$ -measurable graph. Let G be a nonempty and compact valued, continuous correspondence from Z to $\mathcal{M}(X \times S)$. We assume that for any $z \in Z$ and $\tau \in G(z)$, the marginal of τ on S is λ and $\tau(\text{Gr}(A(z, \cdot))) = 1$. Let F be a measurable, nonempty, convex and compact valued correspondence from $\text{Gr}(A) \rightarrow \mathcal{M}(Y)$ such that F is sectionally continuous on $Z \times X$. Define a correspondence Π from Z to $\mathcal{M}(X \times S \times Y)$ by letting*

$$\begin{aligned} \Pi(z) = \{ & g(z) \diamond f(z, \cdot) : g \text{ is a Borel measurable selection of } G, \\ & f \text{ is a Borel measurable selection of } F \}. \end{aligned}$$

Then the correspondence Π is nonempty and compact valued, and continuous.

Proof. Let \mathcal{S} be the completion of $\mathcal{B}(S)$ under the probability measure λ . By Lemma 12, $\tilde{A}(s) = \text{Gr}(A(s, \cdot))$ can be viewed as an \mathcal{S} -measurable mapping from S to the set of nonempty and compact subsets $\mathcal{K}_{Z \times X}$ of $Z \times X$. For any $s \in S$, the correspondence $F_s = F(\cdot, s)$ is continuous on $\tilde{A}(s)$. By Lemma 4, there exists a measurable, nonempty and compact valued correspondence \tilde{F} from $Z \times X \times S$ to $\mathcal{M}(Y)$ and a Borel measurable subset S' of S with $\lambda(S') = 1$ such that for each $s \in S'$, \tilde{F}_s is continuous on $Z \times X$, and the restriction of \tilde{F}_s to $\tilde{A}(s)$ is F_s .

By Lemma 4 (Lusin's theorem), there exists a compact subset $S_1 \subseteq S'$ such that \tilde{A} is continuous on S_1 and $\lambda(S_1) > \frac{1}{2}$. Let $K_1 = \tilde{A}(S_1)$. Then $K_1 \subseteq Z \times X$ is compact.

Let $C(K_1, \mathcal{K}_{\mathcal{M}(Y)})$ be the space of continuous functions from K_1 to $\mathcal{K}_{\mathcal{M}(Y)}$, where $\mathcal{K}_{\mathcal{M}(Y)}$ is the set of nonempty and compact subsets of $\mathcal{M}(Y)$. Suppose that the restriction of \mathcal{S} on S_1 is \mathcal{S}_1 . Let \tilde{F}_1 be the restriction of \tilde{F} to $K_1 \times S_1$. Then \tilde{F}_1 can be viewed as an \mathcal{S}_1 -measurable function from S_1 to $C(K_1, \mathcal{K}_{\mathcal{M}(Y)})$ (see Theorem 4.55 in Aliprantis and Border (2006)). Again by Lemma 4 (Lusin's theorem), there exists a compact subset of S_1 , say itself, such that $\lambda(S_1) > \frac{1}{2}$ and \tilde{F}_1 is continuous on S_1 . As a result, \tilde{F}_1 is a continuous correspondence on $\text{Gr}(A) \cap (S_1 \times Z \times X)$, so is F . Let λ_1 be a probability measure on S_1 such that $\lambda_1(D) = \frac{\lambda(D)}{\lambda(S_1)}$ for any measurable subset $D \subseteq S_1$.

For any $z \in Z$ and $\tau \in G(z)$, the definition of G implies that there exists a transition probability ν from S to X such that $\lambda \diamond \nu = \tau$. Define a correspondence G_1 from Z to $\mathcal{M}(X \times S)$ as follows: for any $z \in Z$, $G_1(z)$ is the set of all $\tau_1 = \lambda_1 \diamond \nu$ such that $\tau = \lambda \diamond \nu \in G(z)$. It can be easily checked that G_1 is also a nonempty and compact valued, and continuous correspondence. Let

$$\begin{aligned} \Pi_1(z) &= \{\tau_1 \diamond f(z, \cdot) : \tau_1 = \lambda_1 \diamond \nu \in G_1(z), \\ &\quad f \text{ is a Borel measurable selection of } \tilde{F}\}. \end{aligned}$$

By Lemma 10, Π_1 is nonempty and compact valued, and continuous. Furthermore, it is easy to see that for any z , $\Pi_1(z)$ coincides with the set

$$\{(\lambda_1 \diamond \nu) \diamond f(z, \cdot) : \lambda \diamond \nu \in G(z), f \text{ is a Borel measurable selection of } F\}.$$

Repeat this procedure, one can find a sequence of compact subsets $\{S_t\}$ such that (1) for any $t \geq 1$, $S_t \subseteq S'$, $S_t \cap (S_1 \cup \dots \cup S_{t-1}) = \emptyset$ and $\lambda(S_1 \cup \dots \cup S_t) \geq \frac{t}{t+1}$, (2) F is continuous on $\text{Gr}(A) \cap (S_t \times Z \times X)$, λ_t is a probability measure on S_t such that $\lambda_t(D) = \frac{\lambda(D)}{\lambda(S_t)}$ for any measurable subset $D \subseteq S_t$, and (3) the correspondence

$$\Pi_t(z) = \{(\lambda_t \diamond \nu) \diamond f(z, \cdot) : \lambda \diamond \nu \in G(z),$$

f is a Borel measurable selection of F }.

is nonempty and compact valued, and continuous.

Pick a sequence $\{z_k\}$, $\{\nu_k\}$ and $\{f_k\}$ such that $(\lambda \diamond \nu_k) \diamond f_k(z_k, \cdot) \in \Pi(z_k)$, $z_k \rightarrow z_0$ and $(\lambda \diamond \nu_k) \diamond f_k(z_k, \cdot)$ weakly converges to some κ . It is easy to see that $(\lambda_t \diamond \nu_k) \diamond f_k(z_k, \cdot) \in \Pi_t(z_k)$ for each t . As Π_1 is compact valued and continuous, it has a subsequence, say itself, such that z_k converges to some $z_0 \in Z$ and $(\lambda_1 \diamond \nu_k) \diamond f_k(z_k, \cdot)$ weakly converges to some $(\lambda_1 \diamond \mu^1) \diamond f^1(z_0, \cdot) \in \Pi_1(z_0)$. Repeat this procedure, one can get a sequence of $\{\mu^m\}$ and f^m . Let $\mu(s) = \mu^m(s)$ and $f(z_0, s, x) = f^m(z_0, s, x)$ for any $x \in A(z_0, s)$ when $s \in S_m$. By Lemma 13, $(\lambda \diamond \mu) \diamond f(z_0, \cdot) = \kappa$, which implies that Π is upper hemicontinuous.

Similarly, the compactness and lower hemicontinuity of Π follow from the compactness and lower hemicontinuity of Π_t for each t . \square

The next lemma presents the convergence property for the integrals of a sequence of functions and probability measures.

Lemma 15. *Let S and X be Polish spaces, and A a measurable, nonempty and compact valued correspondence from S to X . Suppose that λ is a Borel probability measure on S and $\{\nu_n\}_{1 \leq n \leq \infty}$ is a sequence of transition probabilities from S to $\mathcal{M}(X)$ such that $\nu_n(A(s)|s) = 1$ for each s and n . For each $n \geq 1$, let $\tau_n = \lambda \diamond \nu_n$. Assume that the sequence $\{\tau_n\}$ of Borel probability measures on $S \times X$ converges weakly to a Borel probability measure τ_∞ on $S \times X$. Let $\{g_n\}_{1 \leq n \leq \infty}$ be a sequence of functions satisfying the following three properties.*

1. *For each n between 1 and ∞ , $g_n: S \times X \rightarrow \mathbb{R}_+$ is measurable and sectionally continuous on X .*
2. *For any $s \in S$ and any sequence $x_n \rightarrow x_\infty$ in X , $g_n(s, x_n) \rightarrow g_\infty(s, x_\infty)$ as $n \rightarrow \infty$.*
3. *The sequence $\{g_n\}_{1 \leq n \leq \infty}$ is integrably bounded in the sense that there exists a λ -integrable function $\psi: S \rightarrow \mathbb{R}_+$ such that for any n , s and x , $g_n(s, x) \leq \psi(s)$.*

Then we have

$$\int_{S \times X} g_n(s, x) \tau_n(d(s, x)) \rightarrow \int_{S \times X} g_\infty(s, x) \tau_\infty(d(s, x)).$$

Proof. By Theorem 2.1.3 in [Castaing, De Fitte and Valadier \(2004\)](#), for any integrably bounded function $g: S \times X \rightarrow \mathbb{R}_+$ which is sectionally continuous on X , we have

$$\int_{S \times X} g(s, x) \tau_n(d(s, x)) \rightarrow \int_{S \times X} g(s, x) \tau_\infty(d(s, x)). \quad (2)$$

Let $\{y_n\}_{1 \leq n \leq \infty}$ be a sequence such that $y_n = \frac{1}{n}$ and $y_\infty = 0$. Then $y_n \rightarrow y_\infty$. Define a mapping \tilde{g} from $S \times X \times \{y_1, \dots, y_\infty\}$ such that $\tilde{g}(s, x, y_n) = g_n(s, x)$. Then \tilde{g} is measurable on S and continuous on $X \times \{y_1, \dots, y_\infty\}$. Define a correspondence G from S to $X \times \{y_1, \dots, y_\infty\} \times \mathbb{R}_+$ such that

$$G(s) = \{(x, y_n, c) : c \in \tilde{g}(s, x, y_n), x \in A(s), 1 \leq n \leq \infty\}.$$

For any s , $A(s) \times \{y_1, \dots, y_\infty\}$ is compact and $\tilde{g}(s, \cdot, \cdot)$ is continuous. By Lemma 3 (6), $G(s)$ is compact. By Lemma 2 (2), G can be viewed as a measurable mapping from S to the space of nonempty compact subsets of $X \times \{y_1, \dots, y_\infty\} \times \mathbb{R}_+$. Similarly, A can be viewed as a measurable mapping from S to the space of nonempty compact subsets of X .

Fix an arbitrary $\epsilon > 0$. By Lemma 4 (Lusin's theorem), there exists a compact subset $S_1 \subseteq S$ such that A and G are continuous on S_1 and $\lambda(S \setminus S_1) < \epsilon$. Without loss of generality, we can assume that $\lambda(S \setminus S_1)$ is sufficiently small such that $\int_{S \setminus S_1} \psi(s) \lambda(ds) < \frac{\epsilon}{6}$. Thus, for any n ,

$$\int_{(S \setminus S_1) \times X} \psi(s) \tau_n(d(s, x)) = \int_{(S \setminus S_1)} \psi(s) \nu_n(X) \lambda(ds) < \frac{\epsilon}{6}.$$

By Lemma 3 (6), the set $E = \{(s, x) : s \in S_1, x \in A(s)\}$ is compact. Since G is continuous on S_1 , \tilde{g} is continuous on $E \times \{y_1, \dots, y_\infty\}$. Since $E \times \{y_1, \dots, y_\infty\}$ is compact, \tilde{g} is uniformly continuous on $E \times \{y_1, \dots, y_\infty\}$. Thus, there exists a positive integer $N_1 > 0$ such that for any $n \geq N_1$, $|g_n(s, x) - g_\infty(s, x)| < \frac{\epsilon}{3}$ for any $(s, x) \in E$.

By Equation (2), there exists a positive integer N_2 such that for any $n \geq N_2$,

$$\left| \int_{S \times X} g_\infty(s, x) \tau_n(d(s, x)) - \int_{S \times X} g_\infty(s, x) \tau_\infty(d(s, x)) \right| < \frac{\epsilon}{3}.$$

Let $N_0 = \max\{N_1, N_2\}$. For any $n \geq N_0$,

$$\begin{aligned} & \left| \int_{S \times X} g_n(s, x) \tau_n(d(s, x)) - \int_{S \times X} g_\infty(s, x) \tau_\infty(d(s, x)) \right| \\ & \leq \left| \int_{S \times X} g_n(s, x) \tau_n(d(s, x)) - \int_{S \times X} g_\infty(s, x) \tau_n(d(s, x)) \right| \\ & + \left| \int_{S \times X} g_\infty(s, x) \tau_n(d(s, x)) - \int_{S \times X} g_\infty(s, x) \tau_\infty(d(s, x)) \right| \\ & \leq \left| \int_{S_1 \times X} g_n(s, x) \tau_n(d(s, x)) - \int_{S_1 \times X} g_\infty(s, x) \tau_n(d(s, x)) \right| \\ & + \left| \int_{(S \setminus S_1) \times X} g_n(s, x) \tau_n(d(s, x)) - \int_{(S \setminus S_1) \times X} g_\infty(s, x) \tau_n(d(s, x)) \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \int_{S \times X} g_\infty(s, x) \tau_n(d(s, x)) - \int_{S \times X} g_\infty(s, x) \tau_\infty(d(s, x)) \right| \\
& \leq \int_E |g_n(s, x) - g_\infty(s, x)| \tau_n(d(s, x)) + 2 \cdot \int_{(S \setminus S_1) \times X} \psi(s) \tau_n(d(s, x)) \\
& + \left| \int_{S \times X} g_\infty(s, x) \tau_n(d(s, x)) - \int_{S \times X} g_\infty(s, x) \tau_\infty(d(s, x)) \right| \\
& < \frac{\epsilon}{3} + 2 \cdot \frac{\epsilon}{6} + \frac{\epsilon}{3} \\
& = \epsilon.
\end{aligned}$$

This completes the proof. □

5.2 Discontinuous games with endogenous stochastic sharing rules

[Simon and Zame \(1990\)](#) proved the existence of a Nash equilibrium in discontinuous games with endogenous sharing rules. In particular, they considered a static game with a payoff correspondence P that is bounded, nonempty, convex and compact valued, and upper hemicontinuous. They showed that there exists a Borel measurable selection p of the payoff correspondence, namely the endogenous sharing rule, and a mixed strategy profile α such that α is a Nash equilibrium when players take p as the payoff function.

In this subsection, we shall consider discontinuous games with endogenous stochastic sharing rules. That is, we allow the payoff correspondence to depend on some state variable in a measurable way as follows:

1. let S be a Borel subset of a Polish space, Y a Polish space, and λ a Borel probability measure on S ;
2. D is a $\mathcal{B}(S) \otimes \mathcal{B}(Y)$ -measurable subset of $S \times Y$, where $D(s)$ is compact for all $s \in S$ and $\lambda(\{s \in S : D(s) \neq \emptyset\}) > 0$;
3. $X = \prod_{1 \leq i \leq n} X_i$, where each X_i is a Polish space;
4. for each i , A_i is a measurable, nonempty and compact valued correspondence from D to X_i , which is sectionally continuous on Y ;
5. $A = \prod_{1 \leq i \leq n} A_i$ and $E = \text{Gr}(A)$;
6. P is a bounded, measurable, nonempty, convex and compact valued correspondence from E to \mathbb{R}^n which is essentially sectionally upper hemicontinuous on $Y \times X$.

A stochastic sharing rule at $(s, y) \in D$ is a Borel measurable selection of the correspondence $P(s, y, \cdot)$; i.e., a Borel measurable function $p: A(s, y) \rightarrow \mathbb{R}^n$ such

that $p(x) \in P(s, y, x)$ for all $x \in A(s, y)$. Given $(s, y) \in D$, $P(s, y, \cdot)$ represents the set of all possible payoff profiles, and a sharing rule p is a particular choice of the payoff profile.

Now we shall prove the following proposition.

Proposition 4. *There exists a $\mathcal{B}(D)$ -measurable, nonempty and compact valued correspondence Φ from D to $\mathbb{R}^n \times \mathcal{M}(X) \times \Delta(X)$ such that Φ is essentially sectionally upper hemicontinuous on Y , and for λ -almost all $s \in S$ with $D(s) \neq \emptyset$ and $y \in D(s)$, $\Phi(s, y)$ is the set of points (v, α, μ) that*

1. $v = \int_X p(s, y, x) \alpha(dx)$ such that $p(s, y, \cdot)$ is a Borel measurable selection of $P(s, y, \cdot)$;²⁹
2. $\alpha \in \otimes_{i \in I} \mathcal{M}(A_i(s, y))$ is a Nash equilibrium in the subgame (s, y) with payoff $p(s, y, \cdot)$ and action space $A_i(s, y)$ for each player i ;
3. $\mu = p(s, y, \cdot) \circ \alpha$.³⁰

In addition, denote the restriction of Φ on the first component \mathbb{R}^n as $\Phi|_{\mathbb{R}^n}$, which is a correspondence from D to \mathbb{R}^n . Then $\Phi|_{\mathbb{R}^n}$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on Y .

If D is a closed subset, P is upper hemicontinuous on E and A_i is continuous on D for each $i \in I$, then Proposition 4 is reduced to be the following lemma (see [Simon and Zame \(1990\)](#) and [Reny and Robson \(2002, Lemma 4\)](#)).

Lemma 16. *Assume that D is a closed subset, P is upper hemicontinuous on E and A_i is continuous on D for each $i \in I$. Consider the correspondence $\Phi: D \rightarrow \mathbb{R}^n \times \mathcal{M}(X) \times \Delta(X)$ defined as follows: $(v, \alpha, \mu) \in \Phi(s, y)$ if*

1. $v = \int_X p(s, y, x) \alpha(dx)$ such that $p(s, y, \cdot)$ is a Borel measurable selection of $P(s, y, \cdot)$;
2. $\alpha \in \otimes_{i \in I} \mathcal{M}(A_i(s, y))$ is a Nash equilibrium in the subgame (s, y) with payoff $p(s, y, \cdot)$ and action space $A_i(s, y)$ for each player i ;
3. $\mu = p(s, y, \cdot) \circ \alpha$.

Then Φ is nonempty and compact valued, and upper hemicontinuous on D .

We shall now prove Proposition 4.

Proof of Proposition 4. There exists a Borel subset $\hat{S} \subseteq S$ with $\lambda(\hat{S}) = 1$ such that $D(s) \neq \emptyset$ for each $s \in \hat{S}$, and P is sectionally upper hemicontinuous on Y when it is restricted on $D \cap (\hat{S} \times Y)$. Without loss of generality, we assume that $\hat{S} = S$.

²⁹Note that we require $p(s, y, \cdot)$ to be measurable for each (s, y) , but p may not be jointly measurable.

³⁰The finite measure $\mu = p(s, y, \cdot) \circ \alpha$ if $\mu(B) = \int_B p(s, y, x) \alpha(dx)$ for any Borel subset $B \subseteq X$.

Suppose that \mathcal{S} is the completion of $\mathcal{B}(S)$ under the probability measure λ . Let \mathcal{D} and \mathcal{E} be the restrictions of $\mathcal{S} \otimes \mathcal{B}(Y)$ and $\mathcal{S} \otimes \mathcal{B}(Y) \otimes \mathcal{B}(X)$ on D and E , respectively.

Define a correspondence \tilde{D} from S to Y such that $\tilde{D}(s) = \{y \in Y : (s, y) \in D\}$. Then \tilde{D} is nonempty and compact valued. By Lemma 2 (4), \tilde{D} is \mathcal{S} -measurable.

Since $\tilde{D}(s)$ is compact and $A(s, \cdot)$ is upper hemicontinuous for any $s \in S$, $E(s)$ is compact by Lemma 3 (6). Define a correspondence Γ from S to $Y \times X \times \mathbb{R}^n$ as $\Gamma(s) = \text{Gr}(P(s, \cdot, \cdot))$. For all s , $P(s, \cdot, \cdot)$ is bounded, upper hemicontinuous and compact valued on $E(s)$, hence it has a compact graph. As a result, Γ is compact valued. By Lemma 2 (1), P has an $\mathcal{S} \otimes \mathcal{B}(Y \times X \times \mathbb{R}^n)$ -measurable graph. Since $\text{Gr}(\Gamma) = \text{Gr}(P)$, $\text{Gr}(\Gamma)$ is $\mathcal{S} \otimes \mathcal{B}(Y \times X \times \mathbb{R}^n)$ -measurable. Due to Lemma 2 (4), the correspondence Γ is \mathcal{S} -measurable. We can view Γ as a function from S into the space \mathcal{K} of nonempty compact subsets of $Y \times X \times \mathbb{R}^n$. By Lemma 1, \mathcal{K} is a Polish space endowed with the Hausdorff metric topology. Then by Lemma 2 (2), Γ is an \mathcal{S} -measurable function from S to \mathcal{K} . One can also define a correspondence \tilde{A}_i from S to $Y \times X$ as $\tilde{A}_i(s) = \text{Gr}(A_i(s, \cdot))$. It is easy to show that \tilde{A}_i can be viewed as an \mathcal{S} -measurable function from S to the space of nonempty compact subsets of $Y \times X$, which is endowed with the Hausdorff metric topology. By a similar argument, \tilde{D} can be viewed as an \mathcal{S} -measurable function from S to the space of nonempty compact subsets of Y .

By Lemma 4 (Lusin's Theorem), there exists a compact subset $S_1 \subseteq S$ such that $\lambda(S \setminus S_1) < \frac{1}{2}$, Γ , \tilde{D} and $\{\tilde{A}_i\}_{1 \leq i \leq n}$ are continuous functions on S_1 . By Lemma 2 (3), Γ , \tilde{D} and \tilde{A}_i are continuous correspondences on S_1 . Let $D_1 = \{(s, y) \in D : s \in S_1, y \in \tilde{D}(s)\}$. Since S_1 is compact and \tilde{D} is continuous, D_1 is compact (see Lemma 3 (6)). Similarly, $E_1 = E \cap (S_1 \times Y \times X)$ is also compact. Thus, P is an upper hemicontinuous correspondence on E_1 . Define a correspondence Φ_1 from D_1 to $\mathbb{R}^n \times \mathcal{M}(X) \times \Delta(X)$ as in Lemma 16, then it is nonempty and compact valued, and upper hemicontinuous on D_1 .

Following the same procedure, for any $m \geq 1$, there exists a compact subset $S_m \subseteq S$ such that (1) $S_m \cap (\cup_{1 \leq k \leq m-1} S_k) = \emptyset$ and $D_m = D \cap (S_m \times Y)$ is compact, (2) $\lambda(S_m) > 0$ and $\lambda(S \setminus (\cup_{1 \leq k \leq m} S_k)) < \frac{1}{2m}$, and (3) there is a nonempty and compact valued, upper hemicontinuous correspondence Φ_m from D_m to $\mathbb{R}^n \times \mathcal{M}(X) \times \Delta(X)$, which satisfies conditions (1)-(3) in Lemma 16. Thus, we have countably many disjoint sets $\{S_m\}_{m \geq 1}$ such that (1) $\lambda(\cup_{m \geq 1} S_m) = 1$, (2) Φ_m is nonempty and compact valued, and upper hemicontinuous on each D_m , $m \geq 1$.

Since A_i is a $\mathcal{B}(S) \otimes \mathcal{B}(Y)$ -measurable, nonempty and compact valued correspondence, it has a Borel measurable selection a_i by Lemma 3 (3). Fix a Borel measurable selection p of P (such a selection exists also due to Lemma 3 (3)).

Define a mapping (v_0, α_0, μ_0) from D to $\mathbb{R}^n \times \mathcal{M}(X) \times \Delta(X)$ such that (1) $\alpha_i(s, y) = \delta_{a_i(s, y)}$ and $\alpha_0(s, y) = \otimes_{i \in I} \alpha_i(s, y)$; (2) $v_0(s, y) = p(s, y, a_1(s, y), \dots, a_n(s, y))$ and (3) $\mu_0(s, y) = p(s, y, \cdot) \circ \alpha_0$. Let $D_0 = D \setminus (\cup_{m \geq 1} D_m)$ and $\Phi_0(s, y) = \{(v_0(s, y), \alpha_0(s, y), \mu_0(s, y))\}$ for $(s, y) \in D_0$. Then, Φ_0 is $\mathcal{B}(S) \otimes \mathcal{B}(Y)$ -measurable, nonempty and compact valued.

Let $\Phi(s, y) = \Phi_m(s, y)$ if $(s, y) \in D_m$ for some $m \geq 0$. Then, $\Phi(s, y)$ satisfies conditions (1)-(3) if $(s, y) \in D_m$ for $m \geq 1$. That is, Φ is $\mathcal{B}(D)$ -measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on Y , and satisfies conditions (1)-(3) for λ -almost all $s \in S$.

Then consider $\Phi|_{\mathbb{R}^n}$, which is the restriction of Φ on the first component \mathbb{R}^n . Let $\Phi_m|_{\mathbb{R}^n}$ be the restriction of Φ_m on the first component \mathbb{R}^n with the domain D_m for each $m \geq 0$. It is obvious that $\Phi_0|_{\mathbb{R}^n}$ is measurable, nonempty and compact valued. For each $m \geq 1$, D_m is compact and Φ_m is upper hemicontinuous and compact valued. By Lemma 3 (6), $\text{Gr}(\Phi_m)$ is compact. Thus, $\text{Gr}(\Phi_m|_{\mathbb{R}^n})$ is also compact. By Lemma 3 (4), $\Phi_m|_{\mathbb{R}^n}$ is measurable. In addition, $\Phi_m|_{\mathbb{R}^n}$ is nonempty and compact valued, and upper hemicontinuous on D_m . Notice that $\Phi|_{\mathbb{R}^n}(s, y) = \Phi_m|_{\mathbb{R}^n}(s, y)$ if $(s, y) \in D_m$ for some $m \geq 0$. Thus, $\Phi|_{\mathbb{R}^n}$ is measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on Y .

The proof is complete. \square

5.3 Proofs of Theorem 1 and Proposition 1

5.3.1 Backward induction

For any $t \geq 1$, suppose that the correspondence Q_{t+1} from H_t to \mathbb{R}^n is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^t . For any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$, let

$$\begin{aligned} P_t(h_{t-1}, x_t) &= \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t | h_{t-1}) \\ &= \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) \varphi_{t0}(h_{t-1}, s_t) \lambda_t(ds_t). \end{aligned}$$

It is obvious that the correspondence P_t is measurable and nonempty valued. Since Q_{t+1} is bounded, P_t is bounded. For λ^t -almost all $s^t \in S^t$, $Q_{t+1}(\cdot, s^t)$ is bounded and upper hemicontinuous on $H_t(s^t)$, and $\varphi_{t0}(s^t, \cdot)$ is continuous on $\text{Gr}(A_0^t)(s^t)$. As φ_{t0} is integrably bounded, $P_t(s^{t-1}, \cdot)$ is also upper hemicontinuous on $\text{Gr}(A^t)(s^{t-1})$ for λ^{t-1} -almost all $s^{t-1} \in S^{t-1}$ (see Lemma 5); that is, the correspondence P_t is essentially sectionally upper hemicontinuous on X^t . Again by Lemma 5, P_t is convex and compact valued since λ_t is an atomless probability measure. That is, $P_t: \text{Gr}(A^t) \rightarrow \mathbb{R}^n$ is a bounded, measurable, nonempty, convex and compact valued

correspondence which is essentially sectionally upper hemicontinuous on X^t .

By Proposition 4, there exists a bounded, measurable, nonempty and compact valued correspondence Φ_t from H_{t-1} to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \Delta(X_t)$ such that Φ_t is essentially sectionally upper hemicontinuous on X^{t-1} , and for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$, $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x) \alpha(dx)$ such that $p_t(h_{t-1}, \cdot)$ is a Borel measurable selection of $P_t(h_{t-1}, \cdot)$;
2. $\alpha \in \otimes_{i \in I} \mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame h_{t-1} with payoff $p_t(h_{t-1}, \cdot)$ and action space $\prod_{i \in I} A_{ti}(h_{t-1})$;
3. $\mu = p_t(h_{t-1}, \cdot) \circ \alpha$.

Denote the restriction of Φ_t on the first component \mathbb{R}^n as $\Phi(Q_{t+1})$, which is a correspondence from H_{t-1} to \mathbb{R}^n . By Proposition 4, $\Phi(Q_{t+1})$ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1} .

5.3.2 Forward induction

The following proposition presents the result on the step of forward induction.

Proposition 5. *For any $t \geq 1$ and any Borel measurable selection q_t of $\Phi(Q_{t+1})$, there exists a Borel measurable selection q_{t+1} of Q_{t+1} and a Borel measurable mapping $f_t: H_{t-1} \rightarrow \otimes_{i \in I} \mathcal{M}(X_{ti})$ such that for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$,*

1. $f_t(h_{t-1}) \in \otimes_{i \in I} \mathcal{M}(A_{ti}(h_{t-1}))$;
2. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}) f_t(dx_t|h_{t-1})$;
3. $f_t(\cdot|h_{t-1})$ is a Nash equilibrium in the subgame h_{t-1} with action spaces $A_{ti}(h_{t-1}), i \in I$ and the payoff functions

$$\int_{S_t} q_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(ds_t|h_{t-1}).$$

Proof. We divide the proof into three steps. In step 1, we show that there exist Borel measurable mappings $f_t: H_{t-1} \rightarrow \otimes_{i \in I} \mathcal{M}(X_{ti})$ and $\mu_t: H_{t-1} \rightarrow \Delta(X_t)$ such that (q_t, f_t, μ_t) is a selection of Φ_t . In step 2, we obtain a Borel measurable selection g_t of P_t such that for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$,

1. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} g_t(h_{t-1}, x) f_t(dx|h_{t-1})$;
2. $f_t(h_{t-1})$ is a Nash equilibrium in the subgame h_{t-1} with payoff $g_t(h_{t-1}, \cdot)$ and action space $A_t(h_{t-1})$;

In step 3, we show that there exists a Borel measurable selection q_{t+1} of Q_{t+1} such that for all $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$,

$$g_t(h_{t-1}, x_t) = \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t | h_{t-1}).$$

Combining Steps 1-3, the proof is complete.

Step 1. Let $\Psi_t: \text{Gr}(\Phi_t(Q_{t+1})) \rightarrow \mathcal{M}(X_t) \times \Delta(X_t)$ be

$$\Psi_t(h_{t-1}, v) = \{(\alpha, \mu): (v, \alpha, \mu) \in \Phi_t(h_{t-1})\}.$$

Recall the construction of Φ_t and the proof of Proposition 4, H_{t-1} can be divided into countably many Borel subsets $\{H_{t-1}^m\}_{m \geq 0}$ such that

1. $H_{t-1} = \cup_{m \geq 0} H_{t-1}^m$ and $\frac{\lambda^{t-1}(\cup_{m \geq 1} \text{proj}_{S^{t-1}}(H_{t-1}^m))}{\lambda^{t-1}(\text{proj}_{S^{t-1}}(H_{t-1}))} = 1$, where $\text{proj}_{S^{t-1}}(H_{t-1}^m)$ and $\text{proj}_{S^{t-1}}(H_{t-1})$ are projections of H_{t-1}^m and H_{t-1} on S^{t-1} ;
2. for $m \geq 1$, H_{t-1}^m is compact, Φ_t is upper hemicontinuous on H_{t-1}^m , and P_t is upper hemicontinuous on

$$\{(h_{t-1}, x_t): h_{t-1} \in H_{t-1}^m, x_t \in A_t(h_{t-1})\};$$

3. there exists a Borel measurable mapping (v_0, α_0, μ_0) from H_{t-1}^0 to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \Delta(X_t)$ such that $\Phi_t(h_{t-1}) \equiv \{(v_0(h_{t-1}), \alpha_0(h_{t-1}), \mu_0(h_{t-1}))\}$ for any $h_{t-1} \in H_{t-1}^0$.

Denote the restriction of Φ_t on H_{t-1}^m as Φ_t^m . For $m \geq 1$, $\text{Gr}(\Phi_t^m)$ is compact, and hence the correspondence $\Psi_t^m(h_{t-1}, v) = \{(\alpha, \mu): (v, \alpha, \mu) \in \Phi_t^m(h_{t-1})\}$ has a compact graph. For $m \geq 1$, Ψ_t^m is measurable by Lemma 3 (4), and has a Borel measurable selection ψ_t^m due to Lemma 3 (3). Define $\psi_t^0(h_{t-1}, v_0(h_{t-1})) = (\alpha_0(h_{t-1}), \mu_0(h_{t-1}))$ for $h_{t-1} \in H_{t-1}^0$. For $(h_{t-1}, v) \in \text{Gr}(\Phi(Q_{t+1}))$, let $\psi_t(h_{t-1}, v) = \psi_t^m(h_{t-1}, v)$ if $h_{t-1} \in H_{t-1}^m$. Then ψ_t is a Borel measurable selection of Ψ_t .

Given a Borel measurable selection q_t of $\Phi(Q_{t+1})$, let

$$\phi_t(h_{t-1}) = (q_t(h_{t-1}), \psi_t(h_{t-1}, q_t(h_{t-1}))).$$

Then ϕ_t is a Borel measurable selection of Φ_t . Denote $\tilde{H}_{t-1} = \cup_{m \geq 1} H_{t-1}^m$. By the construction of Φ_t , there exists Borel measurable mappings $f_t: H_{t-1} \rightarrow \otimes_{i \in I} \mathcal{M}(X_{ti})$ and $\mu_t: H_{t-1} \rightarrow \Delta(X_t)$ such that for all $h_{t-1} \in \tilde{H}_{t-1}$,

1. $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x) f_t(dx | h_{t-1})$ such that $p_t(h_{t-1}, \cdot)$ is a Borel measurable selection of $P_t(h_{t-1}, \cdot)$;

2. $f_t(h_{t-1}) \in \otimes_{i \in I} \mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame h_{t-1} with payoff $p_t(h_{t-1}, \cdot)$ and action space $\prod_{i \in I} A_{ti}(h_{t-1})$;
3. $\mu_t(\cdot | h_{t-1}) = p_t(h_{t-1}, \cdot) \circ f_t(\cdot | h_{t-1})$.

Step 2. Since P_t is upper hemicontinuous on $\{(h_{t-1}, x_t) : h_{t-1} \in H_{t-1}^m, x_t \in A_t(h_{t-1})\}$, due to Lemma 7, there exists a Borel measurable mapping g^m such that (1) $g^m(h_{t-1}, x_t) \in P_t(h_{t-1}, x_t)$ for any $h_{t-1} \in H_{t-1}^m$ and $x_t \in A_t(h_{t-1})$, and (2) $g^m(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot | h_{t-1})$ -almost all x_t . Fix an arbitrary Borel measurable selection g' of P_t . Define a Borel measurable mapping from $\text{Gr}(A_t)$ to \mathbb{R}^n as

$$g(h_{t-1}, x_t) = \begin{cases} g^m(h_{t-1}, x_t) & \text{if } h_{t-1} \in H_{t-1}^m \text{ for } m \geq 1; \\ g'(h_{t-1}, x_t) & \text{otherwise.} \end{cases}$$

Then g is a Borel measurable selection of P_t .

In a subgame $h_{t-1} \in \tilde{H}_{t-1}$, let

$$B_{ti}(h_{t-1}) = \{y_i \in A_{ti}(h_{t-1}) :$$

$$\int_{A_{t(-i)}(h_{t-1})} g_i(h_{t-1}, y_i, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)} | h_{t-1}) > \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(dx_t | h_{t-1})\}.$$

Since $g(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot | h_{t-1})$ -almost all x_t ,

$$\int_{A_t(h_{t-1})} g(h_{t-1}, x_t) f_t(dx_t | h_{t-1}) = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x_t) f_t(dx_t | h_{t-1}).$$

Thus, B_{ti} is a measurable correspondence from \tilde{H}_{t-1} to $A_{ti}(h_{t-1})$. Let $B_{ti}^c(h_{t-1}) = A_{ti}(h_{t-1}) \setminus B_{ti}(h_{t-1})$ for each $h_{t-1} \in \tilde{H}_{t-1}$. Then B_{ti}^c is a measurable and closed valued correspondence, which has a Borel measurable graph by Lemma 2. As a result, B_{ti} also has a Borel measurable graph. As $f_t(h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1} \in \tilde{H}_{t-1}$ with payoff $p_t(h_{t-1}, \cdot)$, $f_{ti}(B_{ti}(h_{t-1}) | h_{t-1}) = 0$.

Denote $\beta_i(h_{t-1}, x_t) = \min P_{ti}(h_{t-1}, x_t)$, where $P_{ti}(h_{t-1}, x_t)$ is the projection of $P_t(h_{t-1}, x_t)$ on the i -th dimension. Then the correspondence P_{ti} is measurable and compact valued, and β_i is Borel measurable. Let $\Lambda_i(h_{t-1}, x_t) = \{\beta_i(h_{t-1}, x_t)\} \times [0, \gamma]^{n-1}$, where $\gamma > 0$ is the upper bound of P_t . Denote $\Lambda'_i(h_{t-1}, x_t) = \Lambda_i(h_{t-1}, x_t) \cap P_t(h_{t-1}, x_t)$. Then Λ'_i is a measurable and compact valued correspondence, and hence has a Borel measurable selection β'_i . Note that β'_i is a Borel measurable selection of P_t . Let

$$g_t(h_{t-1}, x_t) =$$

$$\begin{cases} \beta'_i(h_{t-1}, x_t) & \text{if } h_{t-1} \in \tilde{H}_{t-1}, x_{ti} \in B_{ti}(h_{t-1}) \text{ and } x_{tj} \notin B_{tj}(h_{t-1}), \forall j \neq i; \\ g(h_{t-1}, x_t) & \text{otherwise.} \end{cases}$$

Notice that

$$\begin{aligned} & \{(h_{t-1}, x_t) \in \text{Gr}(A_t) : h_{t-1} \in \tilde{H}_{t-1}, x_{ti} \in B_{ti}(h_{t-1}) \text{ and } x_{tj} \notin B_{tj}(h_{t-1}), \forall j \neq i; \} \\ &= \text{Gr}(A_t) \cap \cup_{i \in I} \left((\text{Gr}(B_{ti}) \times \prod_{j \neq i} X_{tj}) \setminus (\cup_{j \neq i} (\text{Gr}(B_{tj}) \times \prod_{k \neq j} X_{tk})) \right), \end{aligned}$$

which is a Borel set. As a result, g_t is a Borel measurable selection of P_t . Moreover, $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for all $h_{t-1} \in \tilde{H}_{t-1}$ and $f_t(\cdot|h_{t-1})$ -almost all x_t .

Fix a subgame $h_{t-1} \in \tilde{H}_{t-1}$. We will show that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $g_t(h_{t-1}, \cdot)$ in the subgame h_{t-1} . Suppose that player i deviates to some action \tilde{x}_{ti} .

If $\tilde{x}_{ti} \in B_{ti}(h_{t-1})$, then player i 's expected payoff is

$$\begin{aligned} & \int_{A_{t(-i)}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ &= \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ &= \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} \beta_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ &\leq \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ &= \int_{A_{t(-i)}(h_{t-1})} p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ &\leq \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(dx_t|h_{t-1}) \\ &= \int_{A_t(h_{t-1})} g_{ti}(h_{t-1}, x_t) f_t(dx_t|h_{t-1}). \end{aligned}$$

The first and the third equalities hold since $f_{tj}(B_{tj}(h_{t-1})|h_{t-1}) = 0$ for each j , and hence $f_{t(-i)}(\prod_{j \neq i} B_{tj}^c(h_{t-1})|h_{t-1}) = f_{t(-i)}(A_{t(-i)}(h_{t-1})|h_{t-1})$. The second equality and the first inequality are due to the fact that $g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = \beta_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = \min P_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) \leq p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)})$ for $x_{t(-i)} \in \prod_{j \neq i} B_{tj}^c(h_{t-1})$. The second inequality holds since $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $p_t(h_{t-1}, \cdot)$ in the subgame h_{t-1} . The fourth equality follows from the fact that $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$ -almost all x_t .

If $\tilde{x}_{ti} \notin B_{ti}(h_{t-1})$, then player i 's expected payoff is

$$\begin{aligned}
& \int_{A_{t(-i)}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\
&= \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\
&= \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\
&= \int_{A_{t(-i)}(h_{t-1})} g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\
&\leq \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(dx_t|h_{t-1}) \\
&= \int_{A_t(h_{t-1})} g_{ti}(h_{t-1}, x_t) f_t(dx_t|h_{t-1}).
\end{aligned}$$

The first and the third equalities hold since

$$f_{t(-i)}\left(\prod_{j \neq i} B_{tj}^c(h_{t-1})|h_{t-1}\right) = f_{t(-i)}(A_{t(-i)}(h_{t-1})|h_{t-1}).$$

The second equality is due to the fact that $g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)})$ for $x_{t(-i)} \in \prod_{j \neq i} B_{tj}^c(h_{t-1})$. The first inequality follows from the definition of B_{ti} , and the fourth equality holds since $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$ -almost all x_t .

Thus, player i cannot improve his payoff in the subgame h_t by a unilateral change in his strategy for any $i \in I$, which implies that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $g_t(h_{t-1}, \cdot)$ in the subgame h_{t-1} .

Step 3. For any $(h_{t-1}, x_t) \in \text{Gr}(A_t)$,

$$P_t(h_{t-1}, x_t) = \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}).$$

By Lemma 6, there exists a Borel measurable mapping q from $\text{Gr}(P_t) \times S_t$ to \mathbb{R}^n such that

1. $q(h_{t-1}, x_t, e, s_t) \in Q_{t+1}(h_{t-1}, x_t, s_t)$ for any $(h_{t-1}, x_t, e, s_t) \in \text{Gr}(P_t) \times S_t$;
2. $e = \int_{S_t} q(h_{t-1}, x_t, e, s_t) f_{t0}(ds_t|h_{t-1})$ for any $(h_{t-1}, x_t, e) \in \text{Gr}(P_t)$, where $(h_{t-1}, x_t) \in \text{Gr}(A_t)$.

Let

$$q_{t+1}(h_{t-1}, x_t, s_t) = q(h_{t-1}, x_t, g_t(h_{t-1}, x_t), s_t)$$

for any $(h_{t-1}, x_t, s_t) \in H_t$. Then q_{t+1} is a Borel measurable selection of Q_{t+1} .

For $(h_{t-1}, x_t) \in \text{Gr}(A_t)$,

$$\begin{aligned} g_t(h_{t-1}, x_t) &= \int_{S_t} q(h_{t-1}, x_t, g_t(h_{t-1}, x_t), s_t) f_{t0}(ds_t | h_{t-1}) \\ &= \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t | h_{t-1}). \end{aligned}$$

Therefore, we have a Borel measurable selection q_{t+1} of Q_{t+1} , and a Borel measurable mapping $f_t: H_{t-1} \rightarrow \otimes_{i \in I} \mathcal{M}(X_{ti})$ such that for all $h_{t-1} \in \tilde{H}_{t-1}$, properties (1)-(3) are satisfied. The proof is complete. \square

If a dynamic game has only T stages for some positive integer $T \geq 1$, then let $Q_{T+1}(h_T) = \{u(h_T)\}$ for any $h_T \in H_T$, and $Q_t = \Phi(Q_{t+1})$ for $1 \leq t \leq T-1$. We can start with the backward induction from the last period and stop at the initial period, then run the forward induction from the initial period to the last period. Thus, the following corollary is immediate.

Corollary 4. *Any finite-horizon dynamic game with the ARM condition has a subgame-perfect equilibrium.*

5.3.3 Infinite horizon case

Pick a sequence $\xi = (\xi_1, \xi_2, \dots)$ such that (1) ξ_m is a transition probability from H_{m-1} to $\mathcal{M}(X_m)$ for any $m \geq 1$, and (2) $\xi_m(A_m(h_{m-1}) | h_{m-1}) = 1$ for any $m \geq 1$ and $h_{m-1} \in H_{m-1}$. Denote the set of all such ξ as Υ .

Fix any $t \geq 1$, define correspondences Ξ_t^t and Δ_t^t as follows: in the subgame h_{t-1} ,

$$\Xi_t^t(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes \lambda_t,$$

and

$$\Delta_t^t(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes f_{t0}(h_{t-1}).$$

For any $m_1 > t$, suppose that the correspondences $\Xi_t^{m_1-1}$ and $\Delta_t^{m_1-1}$ have been defined. Then we can define correspondences $\Xi_t^{m_1}: H_{t-1} \rightarrow \mathcal{M}\left(\prod_{t \leq m \leq m_1} (X_m \times S_m)\right)$ and $\Delta_t^{m_1}: H_{t-1} \rightarrow \mathcal{M}\left(\prod_{t \leq m \leq m_1} (X_m \times S_m)\right)$ as follows:

$$\begin{aligned} \Xi_t^{m_1}(h_{t-1}) &= \{g(h_{t-1}) \diamond (\xi_{m_1}(h_{t-1}, \cdot) \otimes \lambda_{m_1}) : \\ &g \text{ is a Borel measurable selection of } \Xi_t^{m_1-1}, \\ &\xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1})\}, \end{aligned}$$

and

$$\begin{aligned}\Delta_t^{m_1}(h_{t-1}) &= \{g(h_{t-1}) \diamond (\xi_{m_1}(h_{t-1}, \cdot) \otimes f_{m_1,0}(h_{t-1}, \cdot))\}: \\ &g \text{ is a Borel measurable selection of } \Delta_t^{m_1-1}, \\ &\xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1})\},\end{aligned}$$

where $\mathcal{M}(A_{m_1})$ is regarded as a correspondence from H_{m_1-1} to the space of Borel probability measures on X_{m_1} . For any $m_1 \geq t$, let $\rho_{(h_{t-1}, \xi)}^{m_1} \in \Xi_t^{m_1}$ be the probability measure on $\prod_{t \leq m \leq m_1} (X_m \times S_m)$ induced by $\{\lambda_m\}_{t \leq m \leq m_1}$ and $\{\xi_m\}_{t \leq m \leq m_1}$, and $\varrho_{(h_{t-1}, \xi)}^{m_1} \in \Delta_t^{m_1}$ be the probability measure on $\prod_{t \leq m \leq m_1} (X_m \times S_m)$ induced by $\{f_{m,0}\}_{t \leq m \leq m_1}$ and $\{\xi_m\}_{t \leq m \leq m_1}$. Then, $\Xi_t^{m_1}(h_{t-1})$ is the set of all such $\rho_{(h_{t-1}, \xi)}^{m_1}$, while $\Delta_t^{m_1}(h_{t-1})$ is the set of all such $\varrho_{(h_{t-1}, \xi)}^{m_1}$. Note that $\varrho_{(h_{t-1}, \xi)}^{m_1} \in \Delta_t^{m_1}(h_{t-1})$ if and only if $\rho_{(h_{t-1}, \xi)}^{m_1} \in \Xi_t^{m_1}(h_{t-1})$. Both $\varrho_{(h_{t-1}, \xi)}^{m_1}$ and $\rho_{(h_{t-1}, \xi)}^{m_1}$ can be regarded as probability measures on $H_{m_1}(h_{t-1})$.

Similarly, let $\rho_{(h_{t-1}, \xi)}$ be the probability measure on $\prod_{m \geq t} (X_m \times S_m)$ induced by $\{\lambda_m\}_{m \geq t}$ and $\{\xi_m\}_{m \geq t}$, and $\varrho_{(h_{t-1}, \xi)}$ the probability measure on $\prod_{m \geq t} (X_m \times S_m)$ induced by $\{f_{m,0}\}_{m \geq t}$ and $\{\xi_m\}_{m \geq t}$. Denote the correspondence

$$\Xi_t: H_{t-1} \rightarrow \mathcal{M}\left(\prod_{m \geq t} (X_m \times S_m)\right)$$

as the set of all such $\rho_{(h_{t-1}, \xi)}$, and

$$\Delta_t: H_{t-1} \rightarrow \mathcal{M}\left(\prod_{m \geq t} (X_m \times S_m)\right)$$

as the set of all such $\varrho_{(h_{t-1}, \xi)}$.

The following lemma demonstrates the relationship between $\varrho_{(h_{t-1}, \xi)}^{m_1}$ and $\rho_{(h_{t-1}, \xi)}^{m_1}$.

Lemma 17. *For any $m_1 \geq t$ and $h_{t-1} \in H_{t-1}$,*

$$\varrho_{(h_{t-1}, \xi)}^{m_1} = \left(\prod_{t \leq m \leq m_1} \varphi_{m,0}(h_{t-1}, \cdot) \right) \circ \rho_{(h_{t-1}, \xi)}^{m_1}. \quad 31$$

³¹For $m \geq t \geq 1$ and $h_{t-1} \in H_{t-1}$, the function $\varphi_{m,0}(h_{t-1}, \cdot)$ is defined on $H_{m-1}(h_{t-1}) \times S_m$, which is measurable and sectionally continuous on $\prod_{t \leq k \leq m-1} X_k$. By Lemma 4, $\varphi_{m,0}(h_{t-1}, \cdot)$ can be extended to be a measurable function $\dot{\varphi}_{m,0}(h_{t-1}, \cdot)$ on the product space $\left(\prod_{t \leq k \leq m-1} X_k\right) \times \left(\prod_{t \leq k \leq m} S_k\right)$, which is also sectionally continuous on $\prod_{t \leq k \leq m-1} X_k$. Given any $\xi \in \Upsilon$, since $\rho_{(h_{t-1}, \xi)}^m$ concentrates on $H_m(h_{t-1})$, $\varphi_{m,0}(h_{t-1}, \cdot) \circ \rho_{(h_{t-1}, \xi)}^m = \dot{\varphi}_{m,0}(h_{t-1}, \cdot) \circ \rho_{(h_{t-1}, \xi)}^m$. For notational simplicity, we still use $\varphi_{m,0}(h_{t-1}, \cdot)$, instead of $\dot{\varphi}_{m,0}(h_{t-1}, \cdot)$, to denote the above extension. Similarly, we can work with a suitable extension of the payoff function u as needed.

Proof. Fix $\xi \in \Upsilon$, and Borel subsets $C_m \subseteq X_m$ and $D_m \subseteq S_m$ for $m \geq t$. First, we have

$$\begin{aligned} \varrho_{(h_{t-1}, \xi)}^t(C_t \times D_t) &= \xi_t(C_t|h_{t-1}) \cdot f_{t0}(D_t|h_{t-1}) \\ &= \int_{X_t \times S_t} \mathbf{1}_{C_t \times D_t}(x_t, s_t) \varphi_{t0}(h_{t-1}, s_t) (\xi_t(h_{t-1}) \otimes \lambda_t)(d(x_t, s_t)), \end{aligned}$$

which implies that $\varrho_{(h_{t-1}, \xi)}^t = \varphi_{t0}(h_{t-1}, \cdot) \circ \rho_{(h_{t-1}, \xi)}^t$.³²

Suppose that $\varrho_{(h_{t-1}, \xi)}^{m_2} = \left(\prod_{t \leq m \leq m_2} \varphi_{m0}(h_{t-1}, \cdot) \right) \circ \rho_{(h_{t-1}, \xi)}^{m_2}$ for some $m_2 \geq t$. Then

$$\begin{aligned} & \varrho_{(h_{t-1}, \xi)}^{m_2+1} \left(\prod_{t \leq m \leq m_2+1} (C_m \times D_m) \right) \\ &= \varrho_{(h_{t-1}, \xi)}^{m_2} \diamond (\xi_{m_2+1}(h_{t-1}, \cdot) \otimes f_{(m_2+1)0}(h_{t-1}, \cdot)) \left(\prod_{t \leq m \leq m_2+1} (C_m \times D_m) \right) \\ &= \int_{\prod_{t \leq m \leq m_2} (X_m \times S_m)} \int_{X_{m_2+1} \times S_{m_2+1}} \mathbf{1}_{\prod_{t \leq m \leq m_2+1} (C_m \times D_m)}(x_t, \dots, x_{m_2+1}, s_t, \dots, s_{m_2+1}) \cdot \\ & \quad \xi_{m_2+1} \otimes f_{(m_2+1)0}(d(x_{m_2+1}, s_{m_2+1})|h_{t-1}, x_t, \dots, x_{m_2}, s_t, \dots, s_{m_2}) \\ & \quad \varrho_{(h_{t-1}, \xi)}^{m_2}(d(x_t, \dots, x_{m_2}, s_t, \dots, s_{m_2})|h_{t-1}) \\ &= \int_{\prod_{t \leq m \leq m_2} (X_m \times S_m)} \int_{S_{m_2+1}} \int_{X_{m_2+1}} \mathbf{1}_{\prod_{t \leq m \leq m_2+1} (C_m \times D_m)}(x_t, \dots, x_{m_2+1}, s_t, \dots, s_{m_2+1}) \cdot \\ & \quad \varphi_{(m_2+1)0}(h_{t-1}, x_t, \dots, x_{m_2}, s_t, \dots, s_{m_2+1}) \xi_{m_2+1}(dx_{m_2+1}|h_{t-1}, x_t, \dots, x_{m_2}, s_t, \dots, s_{m_2}) \\ & \quad \lambda_{(m_2+1)0}(ds_{m_2+1}) \prod_{t \leq m \leq m_2} \varphi_{m0}(h_{t-1}, x_t, \dots, x_{m-1}, s_t, \dots, s_m) \\ & \quad \rho_{(h_{t-1}, \xi)}^{m_2}(d(x_t, \dots, x_{m_2}, s_t, \dots, s_{m_2})|h_{t-1}) \\ &= \int_{\prod_{t \leq m \leq m_2+1} (X_m \times S_m)} \mathbf{1}_{\prod_{t \leq m \leq m_2+1} (C_m \times D_m)}(x_t, \dots, x_{m_2+1}, s_t, \dots, s_{m_2+1}) \cdot \\ & \quad \prod_{t \leq m \leq m_2+1} \varphi_{m0}(h_{t-1}, x_t, \dots, x_{m-1}, s_t, \dots, s_m) \rho_{(h_{t-1}, \xi)}^{m_2+1}(d(x_t, \dots, x_{m_2}, s_t, \dots, s_{m_2})|h_{t-1}), \end{aligned}$$

which implies that

$$\varrho_{(h_{t-1}, \xi)}^{m_2+1} = \left(\prod_{t \leq m \leq m_2+1} \varphi_{m0}(h_{t-1}, \cdot) \right) \circ \rho_{(h_{t-1}, \xi)}^{m_2+1}.$$

The proof is thus complete. \square

The next lemma shows that the correspondences $\Delta_t^{m_1}$ and Δ_t are nonempty

³²For a set A in a space X , $\mathbf{1}_A$ is the indicator function of A , which is one on A and zero on $X \setminus A$.

and compact valued, and sectionally continuous.

Lemma 18. 1. For any $t \geq 1$, the correspondence $\Delta_t^{m_1}$ is nonempty and compact valued, and sectionally continuous on X^{t-1} for any $m_1 \geq t$.

2. For any $t \geq 1$, the correspondence Δ_t is nonempty and compact valued, and sectionally continuous on X^{t-1} .

Proof. (1) We first show that the correspondence $\Xi_t^{m_1}$ is nonempty and compact valued, and sectionally continuous on X^{t-1} for any $m_1 \geq t$.

Consider the case $m_1 = t \geq 1$, where

$$\Xi_t^t(h_{t-1}) = \mathcal{M}(A_t(h_{t-1})) \otimes \lambda_t.$$

Since A_{ti} is nonempty and compact valued, and sectionally continuous on X^{t-1} , Ξ_t^t is nonempty and compact valued, and sectionally continuous on X^{t-1} .

Now suppose that $\Xi_t^{m_2}$ is nonempty and compact valued, and sectionally continuous on X^{t-1} for some $m_2 \geq t \geq 1$. Notice that

$$\begin{aligned} \Xi_t^{m_2+1}(h_{t-1}) = & \{g(h_{t-1}) \diamond (\xi_{m_2+1}(h_{t-1}, \cdot) \otimes \lambda_{(m_2+1)}) : \\ & g \text{ is a Borel measurable selection of } \Xi_t^{m_2}, \\ & \xi_{m_2+1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_2+1})\}. \end{aligned}$$

First, we claim that $H_t(s_0, s_1, \dots, s_t)$ is compact for any $(s_0, s_1, \dots, s_t) \in S^t$. We prove this claim by induction.

1. Notice that $H_0(s_0) = X_0$ for any $s_0 \in S_0$, which is compact.
2. Suppose that $H_{m'}(s_0, s_1, \dots, s_{m'})$ is compact for some $0 \leq m' \leq t-1$ and any $(s_0, s_1, \dots, s_{m'}) \in S^{m'}$.
3. Since $A_{m'+1}(\cdot, s_0, s_1, \dots, s_{m'})$ is continuous and compact valued, it has a compact graph by Lemma 3 (6), which is $H_{m'+1}(s_0, s_1, \dots, s_{m'+1})$ for any $(s_0, s_1, \dots, s_{m'+1}) \in S^{m'+1}$.

Thus, we prove the claim.

Define a correspondence A_t^t from $H_{t-1} \times S_t$ to X_t as $A_t^t(h_{t-1}, s_t) = A_t(h_{t-1})$. Then A_t^t is nonempty and compact valued, sectionally continuous on X_{t-1} , and has a $\mathcal{B}(X^t \times S^t)$ -measurable graph. Since the graph of $A_t^t(\cdot, s_0, s_1, \dots, s_t)$ is $H_t(s_0, s_1, \dots, s_t)$ and $H_t(s_0, s_1, \dots, s_t)$ is compact, $A_t^t(\cdot, s_0, s_1, \dots, s_t)$ has a compact graph. For any $h_{t-1} \in H_{t-1}$ and $\tau \in \Xi_t^t(h_{t-1})$, the marginal of τ on S_t is λ_t and $\tau(\text{Gr}(A_t^t(h_{t-1}, \cdot))) = 1$.

For any $m_1 > t$, suppose that the correspondence

$$A_t^{m_1-1}: H_{t-1} \times \prod_{t \leq m \leq m_1-1} S_m \rightarrow \prod_{t \leq m \leq m_1-1} X_m$$

has been defined such that

1. it is nonempty and compact valued, sectionally upper hemicontinuous on X_{t-1} , and has a $\mathcal{B}(X^{m_1-1} \times S^{m_1-1})$ -measurable graph;
2. for any $(s_0, s_1, \dots, s_{m_1-1})$, $A_t^{m_1-1}(\cdot, s_0, s_1, \dots, s_{m_1-1})$ has a compact graph;
3. for any $h_{t-1} \in H_{t-1}$ and $\tau \in \Xi_t^{m_1-1}(h_{t-1})$, the marginal of τ on $\prod_{t \leq m \leq m_1-1} S_m$ is $\otimes_{t \leq m \leq m_1-1} \lambda_m$ and $\tau(\text{Gr}(A_t^{m_1-1}(h_{t-1}, \cdot))) = 1$.

We define a correspondence $A_t^{m_1}: H_{t-1} \times \prod_{t \leq m \leq m_1} S_m \rightarrow \prod_{t \leq m \leq m_1} X_m$ as follows:

$$\begin{aligned} A_t^{m_1}(h_{t-1}, s_t, \dots, s_{m_1}) = \{ & (x_t, \dots, x_{m_1}): \\ & x_{m_1} \in A_{m_1}(h_{t-1}, x_t, \dots, x_{m_1-1}, s_t, \dots, s_{m_1-1}), \\ & (x_t, \dots, x_{m_1-1}) \in A_t^{m_1-1}(h_{t-1}, s_t, \dots, s_{m_1-1}) \}. \end{aligned}$$

It is obvious that $A_t^{m_1}$ is nonempty valued. For any $(s_0, s_1, \dots, s_{m_1})$, since $A_t^{m_1-1}(\cdot, s_0, s_1, \dots, s_{m_1-1})$ has a compact graph and $A_{m_1}(\cdot, s_0, s_1, \dots, s_{m_1-1})$ is continuous and compact valued, $A_t^{m_1}(\cdot, s_0, s_1, \dots, s_{m_1})$ has a compact graph by Lemma 3 (6), which implies that $A_t^{m_1}$ is compact valued and sectionally upper hemicontinuous on X_{t-1} . In addition, $\text{Gr}(A_t^{m_1}) = \text{Gr}(A_{m_1}) \times S_{m_1}$, which is $\mathcal{B}(X^{m_1} \times S^{m_1})$ -measurable. For any $h_{t-1} \in H_{t-1}$ and $\tau \in \Xi_t^{m_1}(h_{t-1})$, it is obvious that the marginal of τ on $\prod_{t \leq m \leq m_1} S_m$ is $\otimes_{t \leq m \leq m_1} \lambda_m$ and $\tau(\text{Gr}(A_t^{m_1}(h_{t-1}, \cdot))) = 1$.

By Lemma 14, $\Xi_t^{m_2+1}$ is nonempty and compact valued, and sectionally continuous on X^{t-1} .

Now we show that the correspondence $\Delta_t^{m_1}$ is nonempty and compact valued, and sectionally continuous on X^{t-1} for any $m_1 \geq t$.

Given s^{t-1} and a sequence $\{x_0^k, x_1^k, \dots, x_{t-1}^k\} \in H_{t-1}(s^{t-1})$ for $1 \leq k \leq \infty$. Let $h_{t-1}^k = (s^{t-1}, (x_0^k, x_1^k, \dots, x_{t-1}^k))$. It is obvious that $\Delta_t^{m_1}$ is nonempty valued, we first show that $\Delta_t^{m_1}$ is sectionally upper hemicontinuous on X^{t-1} . Suppose that $\varrho_{(h_{t-1}^k, \xi^k)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^k)$ for $1 \leq k < \infty$ and $(x_0^k, x_1^k, \dots, x_{t-1}^k) \rightarrow (x_0^\infty, x_1^\infty, \dots, x_{t-1}^\infty)$, we need to show that there exists some ξ^∞ such that a subsequence of $\varrho_{(h_{t-1}^k, \xi^k)}^{m_1}$ weakly converges to $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$ and $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^\infty)$.

Since $\Xi_t^{m_1}$ is sectionally upper hemicontinuous on X^{t-1} , there exists some ξ^∞ such that a subsequence of $\rho_{(h_{t-1}^k, \xi^k)}^{m_1}$, say itself, weakly converges to $\rho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$ and

$\rho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Xi_t^{m_1}(h_{t-1}^\infty)$. Then $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^\infty)$.

For any bounded continuous function ψ on $\prod_{t \leq m \leq m_1} (X_m \times S_m)$, let

$$\begin{aligned} \chi_k(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1}) = \\ \psi(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1}) \cdot \prod_{t \leq m \leq m_1} \varphi_{m0}(h_{t-1}^k, x_t, \dots, x_{m-1}, s_t, \dots, s_m). \end{aligned}$$

Then $\{\chi_k\}$ is a sequence of functions satisfying the following three properties.

1. For each k , χ_k is jointly measurable and sectionally continuous on $\prod_{t \leq m \leq m_1} X_m$.
2. For any (s_t, \dots, s_{m_1}) and any sequence $(x_t^k, \dots, x_{m_1}^k) \rightarrow (x_t^\infty, \dots, x_{m_1}^\infty)$ in $\prod_{t \leq m \leq m_1} X_m$, $\chi_k(x_t^k, \dots, x_{m_1}^k, s_t, \dots, s_{m_1}) \rightarrow \chi_\infty(x_t^\infty, \dots, x_{m_1}^\infty, s_t, \dots, s_{m_1})$ as $k \rightarrow \infty$.
3. The sequence $\{\chi_k\}_{1 \leq k \leq \infty}$ is integrably bounded in the sense that there exists a function $\chi': \prod_{t \leq m \leq m_1} S_m \rightarrow \mathbb{R}_+$ such that χ' is $\otimes_{t \leq m \leq m_1} \lambda_m$ -integrable and for any k and $(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1})$, $\chi_k(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1}) \leq \chi'(s_t, \dots, s_{m_1})$.

By Lemma 15, as $k \rightarrow \infty$,

$$\begin{aligned} & \int_{\prod_{t \leq m \leq m_1} (X_m \times S_m)} \chi_k(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1}) \rho_{(h_{t-1}^k, \xi^k)}^{m_1}(\mathfrak{d}(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1})) \\ & \rightarrow \int_{\prod_{t \leq m \leq m_1} (X_m \times S_m)} \chi_\infty(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1}) \rho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}(\mathfrak{d}(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1})). \end{aligned}$$

Then by Lemma 17,

$$\begin{aligned} & \int_{\prod_{t \leq m \leq m_1} (X_m \times S_m)} \psi(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1}) \varrho_{(h_{t-1}^k, \xi^k)}^{m_1}(\mathfrak{d}(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1})) \\ & \rightarrow \int_{\prod_{t \leq m \leq m_1} (X_m \times S_m)} \psi(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1}) \varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}(\mathfrak{d}(x_t, \dots, x_{m_1}, s_t, \dots, s_{m_1})), \end{aligned}$$

which implies that $\varrho_{(h_{t-1}^k, \xi^k)}^{m_1}$ weakly converges to $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$. Therefore, $\Delta_t^{m_1}$ is sectionally upper hemicontinuous on X^{t-1} . If one chooses $h_{t-1}^1 = h_{t-1}^2 = \dots = h_{t-1}^\infty$, then we indeed show that $\Delta_t^{m_1}$ is compact valued.

In the argument above, we indeed proved that if $\rho_{(h_{t-1}^k, \xi^k)}^{m_1}$ weakly converges to $\rho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$, then $\varrho_{(h_{t-1}^k, \xi^k)}^{m_1}$ weakly converges to $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$.

The left is to show that $\Delta_t^{m_1}$ is sectionally lower hemicontinuous on X^{t-1} . Suppose that $(x_0^k, x_1^k, \dots, x_{t-1}^k) \rightarrow (x_0^\infty, x_1^\infty, \dots, x_{t-1}^\infty)$ and $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^\infty)$, we need to show that there exists a subsequence $\{(x_0^{k_m}, x_1^{k_m}, \dots, x_{t-1}^{k_m})\}$ of $\{(x_0^k, x_1^k, \dots, x_{t-1}^k)\}$ and $\varrho_{(h_{t-1}^{k_m}, \xi^{k_m})}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^{k_m})$ for each k_m such that $\varrho_{(h_{t-1}^{k_m}, \xi^{k_m})}^{m_1}$

weakly converges to $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$.

Since $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Delta_t^{m_1}(h_{t-1}^\infty)$, we have $\rho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1} \in \Xi_t^{m_1}(h_{t-1}^\infty)$. Because $\Xi_t^{m_1}$ is sectionally lower hemicontinuous on X^{t-1} , there exists a subsequence of $\{(x_0^k, x_1^k, \dots, x_{t-1}^k)\}$, say itself, and $\rho_{(h_{t-1}^k, \xi^k)}^{m_1} \in \Xi_t^{m_1}(h_{t-1}^k)$ for each k such that $\rho_{(h_{t-1}^k, \xi^k)}^{m_1}$ weakly converges to $\rho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$. As a result, $\varrho_{(h_{t-1}^k, \xi^k)}^{m_1}$ weakly converges to $\varrho_{(h_{t-1}^\infty, \xi^\infty)}^{m_1}$, which implies that $\Delta_t^{m_1}$ is sectionally lower hemicontinuous on X^{t-1} .

Therefore, $\Delta_t^{m_1}$ is nonempty and compact valued, and sectionally continuous on X^{t-1} for any $m_1 \geq t$.

(2) We show that Δ_t is nonempty and compact valued, and sectionally continuous on X^{t-1} .

It is obvious that Δ_t is nonempty valued, we first prove that it is compact valued.

Given h_{t-1} and a sequence $\{\tau^k\} \subseteq \Delta_t(h_{t-1})$, there exists a sequence of $\{\xi^k\}_{k \geq 1}$ such that $\xi^k = (\xi_1^k, \xi_2^k, \dots) \in \Upsilon$ and $\tau^k = \varrho_{(h_{t-1}, \xi^k)}$ for each k .

By (1), Ξ_t^t is compact. Then there exists a measurable mapping g_t such that (1) $g^t = (\xi_1^1, \dots, \xi_{t-1}^1, g_t, \xi_{t+1}^1, \dots) \in \Upsilon$, and (2) a subsequence of $\{\rho_{(h_{t-1}, \xi^k)}^t\}$, say $\{\rho_{(h_{t-1}, \xi^{k_l})}^t\}_{l \geq 1}$, which weakly converges to $\rho_{(h_{t-1}, g^t)}^t$. Note that $\{\xi_{t+1}^k\}$ is a Borel measurable selection of $\mathcal{M}(A_{t+1})$. By Lemma 14, there is a Borel measurable selection g_{t+1} of $\mathcal{M}(A_{t+1})$ such that there is a subsequence of $\{\rho_{(h_{t-1}, \xi^{k_l})}^{t+1}\}_{l \geq 1}$, say $\{\rho_{(h_{t-1}, \xi^{k_{2l}})}^{t+1}\}_{l \geq 1}$, which weakly converges to $\rho_{(h_{t-1}, g^{t+1})}^{t+1}$, where $g^{t+1} = (\xi_1^1, \dots, \xi_{t-1}^1, g_t, g_{t+1}, \xi_{t+2}^1, \dots) \in \Upsilon$.

Repeat this procedure, one can construct a Borel measurable mapping g such that $\rho_{(h_{t-1}, \xi^{k_{11}})}, \rho_{(h_{t-1}, \xi^{k_{22}})}, \rho_{(h_{t-1}, \xi^{k_{33}})}, \dots$ weakly converges to $\rho_{(h_{t-1}, g)}$. That is, $\rho_{(h_{t-1}, g)}$ is a convergent point of $\{\rho_{(h_{t-1}, \xi^k)}\}$, which implies that $\varrho_{(h_{t-1}, g)}$ is a convergent point of $\{\varrho_{(h_{t-1}, \xi^k)}\}$.

The sectional upper hemicontinuity of Δ_t follows a similar argument as above. In particular, given s^{t-1} and a sequence $\{x_0^k, x_1^k, \dots, x_{t-1}^k\} \subseteq H_{t-1}(s^{t-1})$ for $k \geq 0$. Let $h_{t-1}^k = (s^{t-1}, (x_0^k, x_1^k, \dots, x_{t-1}^k))$. Suppose that $(x_0^k, x_1^k, \dots, x_{t-1}^k) \rightarrow (x_0^0, x_1^0, \dots, x_{t-1}^0)$. If $\{\tau^k\} \subseteq \Delta_t(h_{t-1}^k)$ for $k \geq 1$ and $\tau^k \rightarrow \tau^0$, then one can show that $\tau^0 \in \Delta_t(h_{t-1}^0)$ by repeating a similar argument as in the proof above.

Finally, we consider the sectional lower hemicontinuity of Δ_t . Suppose that $\tau^0 \in \Delta_t(h_{t-1}^0)$. Then there exists some $\xi \in \Upsilon$ such that $\tau^0 = \varrho_{(h_{t-1}^0, \xi)}$. Denote $\tilde{\tau}^m = \varrho_{(h_{t-1}^0, \xi)}^m \in \Delta_t^m(h_{t-1}^0)$ for $m \geq t$. As Δ_t^m is continuous, for each m , there exists some $\xi^m \in \Upsilon$ such that $d(\varrho_{(h_{t-1}^0, \xi^m)}^m, \tilde{\tau}^m) \leq \frac{1}{m}$ for k_m sufficiently large, where d is the Prokhorov metric. Let $\tau^m = \varrho_{(h_{t-1}^0, \xi^m)}$. Then τ^m weakly converges to τ^0 , which implies that Δ_t is sectionally lower hemicontinuous. \square

Define a correspondence $Q_t^\tau : H_{t-1} \rightarrow \mathbb{R}_{++}^n$ as follows:

$$Q_t^\tau(h_{t-1}) = \begin{cases} \left\{ \int_{\prod_{m \geq t}(X_m \times S_m)} u(h_{t-1}, x, s) \varrho_{(h_{t-1}, \xi)}(\mathbf{d}(x, s)) : \varrho_{(h_{t-1}, \xi)} \in \Delta_t(h_{t-1}) \right\}; & t > \tau; \\ \Phi(Q_{t+1}^\tau)(h_{t-1}) & t \leq \tau. \end{cases}$$

The lemma below presents several properties of the correspondence Q_t^τ .

Lemma 19. *For any $t, \tau \geq 1$, Q_t^τ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1} .*

Proof. We prove the lemma in three steps.

Step 1. Fix $t > \tau$. We will show that Q_t^τ is bounded, nonempty and compact valued, and sectionally upper hemicontinuous on X^{t-1} .

The boundedness and nonemptiness of Q_t^τ are obvious. We shall prove that Q_t^τ is sectionally upper hemicontinuous on X^{t-1} . Given s^{t-1} and a sequence $\{x_0^k, x_1^k, \dots, x_{t-1}^k\} \subseteq H_{t-1}(s^{t-1})$ for $k \geq 0$. Let $h_{t-1}^k = (s^{t-1}, (x_0^k, x_1^k, \dots, x_{t-1}^k))$. Suppose that $a^k \in Q_t^\tau(h_{t-1}^k)$ for $k \geq 1$, $(x_0^k, x_1^k, \dots, x_{t-1}^k) \rightarrow (x_0^0, x_1^0, \dots, x_{t-1}^0)$ and $a^k \rightarrow a^0$, we need to show that $a^0 \in Q_t^\tau(h_{t-1}^0)$.

By the definition, there exists a sequence $\{\xi^k\}_{k \geq 1}$ such that

$$a^k = \int_{\prod_{m \geq t}(X_m \times S_m)} u(h_{t-1}^k, x, s) \varrho_{(h_{t-1}^k, \xi^k)}(\mathbf{d}(x, s)),$$

where $\xi^k = (\xi_1^k, \xi_2^k, \dots) \in \Upsilon$ for each k . As Δ_t is compact valued and sectionally continuous on X^{t-1} , there exist some $\varrho_{(h_{t-1}^0, \xi^0)} \in \Delta_t(h_{t-1}^0)$ and a subsequence of $\varrho_{(h_{t-1}^k, \xi^k)}$, say itself, which weakly converges to $\varrho_{(h_{t-1}^0, \xi^0)}$ for $\xi^0 = (\xi_1^0, \xi_2^0, \dots) \in \Upsilon$.

We shall show that

$$a^0 = \int_{\prod_{m \geq t}(X_m \times S_m)} u(h_{t-1}^0, x, s) \varrho_{(h_{t-1}^0, \xi^0)}(\mathbf{d}(x, s)).$$

For this aim, we only need to show that for any $\delta > 0$,

$$\left| a^0 - \int_{\prod_{m \geq t}(X_m \times S_m)} u(h_{t-1}^0, x, s) \varrho_{(h_{t-1}^0, \xi^0)}(\mathbf{d}(x, s)) \right| < \delta. \quad (3)$$

Since the game is continuous at infinity, there exists a positive integer $\tilde{M} \geq t$ such that $w^m < \frac{1}{5}\delta$ for any $m > \tilde{M}$.

For each $j > \tilde{M}$, by Lemma 4, there exists a measurable selection ξ'_j of $\mathcal{M}(A_j)$ such that ξ'_j is sectionally continuous on X^{j-1} . Let $\mu : H_{\tilde{M}} \rightarrow \prod_{m > \tilde{M}}(X_m \times S_m)$ be the transition probability which is induced by $(\xi'_{\tilde{M}+1}, \xi'_{\tilde{M}+2}, \dots)$ and

$\{f_{(\tilde{M}+1)0}, f_{(\tilde{M}+2)0}, \dots\}$. By Lemma 10, μ is measurable and sectionally continuous on $X^{\tilde{M}}$. Let

$$V_{\tilde{M}}(h_{t-1}, x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}}) = \int_{\prod_{m>\tilde{M}}(X_m \times S_m)} u(h_{t-1}, x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}}, x, s) d\mu(x, s | h_{t-1}, x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}}).$$

Then $V_{\tilde{M}}$ is bounded and measurable. In addition, $V_{\tilde{M}}$ is sectionally continuous on $X^{\tilde{M}}$ by Lemma 15.

For any $k \geq 0$, we have

$$\begin{aligned} & \left| \int_{\prod_{m \geq t}(X_m \times S_m)} u(h_{t-1}^k, x, s) \varrho_{(h_{t-1}^k, \xi^k)}(d(x, s)) \right. \\ & - \left. \int_{\prod_{t \leq m \leq \tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h_{t-1}^k, x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}}) \varrho_{(h_{t-1}^k, \xi^k)}^{\tilde{M}}(d(x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}})) \right| \\ & \leq w^{\tilde{M}+1} \\ & < \frac{1}{5}\delta. \end{aligned}$$

Since $\varrho_{(h_{t-1}^k, \xi^k)}$ weakly converges to $\varrho_{(h_{t-1}^0, \xi^0)}$ and $\varrho_{(h_{t-1}^k, \xi^k)}^{\tilde{M}}$ is the marginal of $\varrho_{(h_{t-1}^k, \xi^k)}$ on $\prod_{t \leq m \leq \tilde{M}}(X_m \times S_m)$ for any $k \geq 0$, the sequence $\varrho_{(h_{t-1}^k, \xi^k)}^{\tilde{M}}$ also weakly converges to $\varrho_{(h_{t-1}^0, \xi^0)}^{\tilde{M}}$. By Lemma 15, we have

$$\begin{aligned} & \left| \int_{\prod_{t \leq m \leq \tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h_{t-1}^k, x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}}) \varrho_{(h_{t-1}^k, \xi^k)}^{\tilde{M}}(d(x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}})) \right. \\ & - \left. \int_{\prod_{t \leq m \leq \tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h_{t-1}^0, x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}}) \varrho_{(h_{t-1}^0, \xi^0)}^{\tilde{M}}(d(x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}})) \right| \\ & < \frac{1}{5}\delta \end{aligned}$$

for $k \geq K_1$, where K_1 is a sufficiently large positive integer. In addition, there exists a positive integer K_2 such that $|a^k - a^0| < \frac{1}{5}\delta$ for $k \geq K_2$.

Fix $k > \max\{K_1, K_2\}$. Combining the inequalities above, we have

$$\begin{aligned} & \left| \int_{\prod_{m \geq t}(X_m \times S_m)} u(h_{t-1}^0, x, s) \varrho_{(h_{t-1}^0, \xi^0)}(d(x, s)) - a^0 \right| \\ & \leq \left| \int_{\prod_{m \geq t}(X_m \times S_m)} u(h_{t-1}^0, x, s) \varrho_{(h_{t-1}^0, \xi^0)}(d(x, s)) \right. \\ & - \left. \int_{\prod_{t \leq m \leq \tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h_{t-1}^0, x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}}) \varrho_{(h_{t-1}^0, \xi^0)}^{\tilde{M}}(d(x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}})) \right| \\ & + \left| \int_{\prod_{t \leq m \leq \tilde{M}}(X_m \times S_m)} V_{\tilde{M}}(h_{t-1}^0, x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}}) \varrho_{(h_{t-1}^0, \xi^0)}^{\tilde{M}}(d(x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}})) \right| \end{aligned}$$

$$\begin{aligned}
& - \int_{\prod_{t \leq m \leq \tilde{M}} (X_m \times S_m)} V_{\tilde{M}}(h_{t-1}^k, x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}}) \varrho_{(h_{t-1}^k, \xi^k)}^{\tilde{M}}(\mathbf{d}(x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}})) \Big| \\
& + \left| \int_{\prod_{t \leq m \leq \tilde{M}} (X_m \times S_m)} V_{\tilde{M}}(h_{t-1}^k, x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}}) \varrho_{(h_{t-1}^k, \xi^k)}^{\tilde{M}}(\mathbf{d}(x_t, \dots, x_{\tilde{M}}, s_t, \dots, s_{\tilde{M}})) \right. \\
& - \int_{\prod_{m \geq t} (X_m \times S_m)} u(h_{t-1}^k, x, s) \varrho_{(h_{t-1}^k, \xi^k)}(\mathbf{d}(x, s)) \Big| \\
& + \left| \int_{\prod_{m \geq t} (X_m \times S_m)} u(h_{t-1}^k, x, s) \varrho_{(h_{t-1}^k, \xi^k)}(\mathbf{d}(x, s)) - a^0 \right| \\
& < \delta.
\end{aligned}$$

Thus, we proved inequality (3), which implies that Q_t^τ is sectionally upper hemicontinuous on X^{t-1} for $t > \tau$.

Furthermore, to prove that Q_t^τ is compact valued, we only need to consider the case that $\{x_0^k, x_1^k, \dots, x_{t-1}^k\} = \{x_0^0, x_1^0, \dots, x_{t-1}^0\}$ for any $k \geq 0$, and repeat the above proof.

Step 2. Fix $t > \tau$, we will show that Q_t^τ is measurable.

Fix a sequence (ξ'_1, ξ'_2, \dots) , where ξ'_j is a selection of $\mathcal{M}(A_j)$ measurable in s^{j-1} and continuous in x^{j-1} for each j . For any $M \geq t$, let

$$\begin{aligned}
& W_M^M(h_{t-1}, x_t, \dots, x_M, s_t, \dots, s_M) = \\
& \left\{ \int_{\prod_{m > M} (X_m \times S_m)} u(h_{t-1}, x_t, \dots, x_M, s_t, \dots, s_M, x, s) \varrho_{(h_{t-1}, x_t, \dots, x_M, s_t, \dots, s_M, \xi')}(\mathbf{d}(x, s)) \right\}.
\end{aligned}$$

By Lemma 10, $\varrho_{(h_{t-1}, x_t, \dots, x_M, s_t, \dots, s_M, \xi')}$ is measurable from H_M to $\mathcal{M}(\prod_{m > M} (X_m \times S_m))$, and sectionally continuous on X^M . Thus, W_M^M is bounded, measurable, nonempty, convex and compact valued. By Lemma 15, W_M^M is sectionally continuous on X^M .

Suppose that for some $t \leq j \leq M$, W_M^j has been defined such that it is bounded, measurable, nonempty, convex and compact valued, and sectionally continuous on X^j . Let

$$\begin{aligned}
& W_M^{j-1}(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}) = \\
& \left\{ \int_{X_j \times S_j} w_M^j(h_{t-1}, x_t, \dots, x_j, s_t, \dots, s_j) \varrho_{(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}, \xi)}^j(\mathbf{d}(x_j, s_j)) : \right. \\
& \quad \varrho_{(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}, \xi)}^j \in \Delta_j^j(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}), \\
& \quad \left. w_M^j \text{ is a Borel measurable selection of } W_M^j \right\}.
\end{aligned}$$

Let $\check{S}_j = S_j$.³³ Since

$$\begin{aligned} & \int_{X_j \times S_j} W_M^j(h_{t-1}, x_t, \dots, x_j, s_t, \dots, s_j) \varrho_{(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}, \xi)}^j(\mathfrak{d}(x_j, s_j)) \\ &= \int_{S_j} \int_{X_j \times \check{S}_j} W_M^j(h_{t-1}, x_t, \dots, x_j, s_t, \dots, s_j) \rho_{(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}, \xi)}^j(\mathfrak{d}(x_j, \check{s}_j)) \\ & \quad \cdot \varphi_{j0}(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_j) \lambda_j(\mathfrak{d}s_j), \end{aligned}$$

we have

$$\begin{aligned} & W_M^{j-1}(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}) = \\ & \left\{ \int_{S_j} \int_{X_j \times \check{S}_j} w_M^j(h_{t-1}, x_t, \dots, x_j, s_t, \dots, s_j) \rho_{(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}, \xi)}^j(\mathfrak{d}(x_j, \check{s}_j)) \right. \\ & \quad \cdot \varphi_{j0}(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_j) \lambda_j(\mathfrak{d}s_j) : \\ & \quad \rho_{(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}, \xi)}^j \in \Xi_j^j(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}), \\ & \quad \left. w_M^j \text{ is a Borel measurable selection of } W_M^j \right\}. \end{aligned}$$

Let

$$\begin{aligned} & \check{W}_M^j(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_j) = \\ & \left\{ \int_{X_j \times \check{S}_j} w_M^j(h_{t-1}, x_t, \dots, x_j, s_t, \dots, s_j) \cdot \rho_{(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}, \xi)}^j(\mathfrak{d}(x_j, \check{s}_j)) : \right. \\ & \quad \rho_{(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}, \xi)}^j \in \Xi_j^j(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}), \\ & \quad \left. \check{w}_M^j \text{ is a Borel measurable selection of } W_M^j \right\}. \end{aligned}$$

Since $W_M^j(h_{t-1}, x_t, \dots, x_j, s_t, \dots, s_j)$ is continuous in x_j and does not depend on \check{s}_j , it is continuous in (x_j, \check{s}_j) . In addition, W_M^j is bounded, measurable, nonempty, convex and compact valued. By Lemma 11, \check{W}_M^j is bounded, measurable, nonempty and compact valued, and sectionally continuous on X^{j-1} .

It is easy to see that

$$\begin{aligned} & W_M^{j-1}(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_{j-1}) = \\ & \int_{S_j} \check{W}_M^j(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_j) \varphi_{j0}(h_{t-1}, x_t, \dots, x_{j-1}, s_t, \dots, s_j) \lambda_j(\mathfrak{d}s_j). \end{aligned}$$

By Lemma 5, it is bounded, measurable, nonempty and compact valued, and sectionally continuous on X^{j-1} . By induction, one can show that W_M^{t-1} is bounded,

³³We will need to use Lemma 11 below, which requires the continuity of the correspondences in terms of the integrated variables. Since W_M^j is only measurable, but not continuous, in s_j , we add a dummy variable \check{s}_j so that W_M^j is trivially continuous in such a variable.

measurable, nonempty and compact valued, and sectionally continuous on X^{t-1} .

Let $W^{t-1} = \overline{\cup_{M \geq t} W_M^{t-1}}$. That is, W^{t-1} is the closure of $\cup_{M \geq t} W_M^{t-1}$, which is measurable due to Lemma 3.

First, $W^{t-1} \subseteq Q_t^\tau$ because $W_M^{t-1} \subseteq Q_t^\tau$ for each $M \geq t$ and Q_t^τ is compact valued. Second, fix h_{t-1} and $q \in Q_t^\tau(h_{t-1})$. Then there exists a mapping $\xi \in \Upsilon$ such that

$$q = \int_{\prod_{m \geq t} (X_m \times S_m)} u(h_{t-1}, x, s) \varrho_{(h_{t-1}, \xi)}(\mathrm{d}(x, s)).$$

For $M \geq t$, let

$$V_M(h_{t-1}, x_t, \dots, x_M, s_t, \dots, s_M) = \int_{\prod_{m > M} (X_m \times S_m)} u(h_{t-1}, x_t, \dots, x_M, s_t, \dots, s_M, x, s) \varrho_{(h_{t-1}, x_t, \dots, x_M, s_t, \dots, s_M, \xi)}(x, s)$$

and

$$q_M = \int_{\prod_{t \leq m \leq M} (X_m \times S_m)} V_M(h_{t-1}, x, s) \varrho_{(h_{t-1}, \xi)}^M(\mathrm{d}(x, s)).$$

Hence, $q_M \in W_M^{t-1}$. Because the dynamic game is continuous at infinity, $q_M \rightarrow q$, which implies that $q \in W^{t-1}(h_{t-1})$ and $Q_t^\tau \subseteq W^{t-1}$.

Therefore, $W^{t-1} = Q_t^\tau$, and hence Q_t^τ is measurable for $t > \tau$.

Step 3. For $t \leq \tau$, we can start with $Q_{\tau+1}^\tau$. Repeating the backward induction in Subsection 5.3.1, we have that Q_t^τ is also bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1} . \square

Denote

$$Q_t^\infty = \begin{cases} Q_t^{t-1}, & \text{if } \cap_{\tau \geq 1} Q_t^\tau = \emptyset; \\ \cap_{\tau \geq 1} Q_t^\tau, & \text{otherwise.} \end{cases}$$

The following three lemmas show that $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1}) = E_t(h_{t-1})$ for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$.³⁴

Lemma 20. 1. *The correspondence Q_t^∞ is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1} .*

2. *For any $t \geq 1$, $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1})$ for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$.*

Proof. (1) It is obvious that Q_t^∞ is bounded. By the definition of Q_t^τ , for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$, $Q_t^{\tau_1}(h_{t-1}) \subseteq Q_t^{\tau_2}(h_{t-1})$ for $\tau_1 \geq \tau_2$. Since Q_t^τ is nonempty and compact valued, $Q_t^\infty = \cap_{\tau \geq 1} Q_t^\tau$ is nonempty and compact valued for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$. If $\cap_{\tau \geq 1} Q_t^\tau = \emptyset$, then $Q_t^\infty = Q_t^{t-1}$. Thus, $Q_t^\infty(h_{t-1})$ is

³⁴The proofs for Lemmas 20 and 22 follow the standard ideas with various modifications; see, for example, Harris (1990), Harris, Reny and Robson (1995) and Mariotti (2000).

nonempty and compact valued for all $h_{t-1} \in H_{t-1}$. By Lemma 3 (2), $\bigcap_{\tau \geq 1} Q_t^\tau$ is measurable, which implies that Q_t^∞ is measurable.

Fix any $s^{t-1} \in S^{t-1}$ such that $Q_t^\tau(\cdot, s^{t-1})$ is upper hemicontinuous on $H_{t-1}(s^{t-1})$ for any τ . By Lemma 3 (7), $Q_t^\tau(\cdot, s^{t-1})$ has a closed graph for each τ , which implies that $Q_t^\infty(\cdot, s^{t-1})$ has a closed graph. Referring to Lemma 3 (7) again, $Q_t^\infty(\cdot, s^{t-1})$ is upper hemicontinuous on $H_{t-1}(s^{t-1})$. Since Q_t^τ is essentially upper hemicontinuous on X^{t-1} for each τ , Q_t^∞ is essentially upper upper hemicontinuous on X^{t-1} .

(2) For any $\tau \geq 1$ and λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$, $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq \Phi(Q_{t+1}^\tau)(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$, and hence $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq Q_t^\infty(h_{t-1})$.

The space $\{1, 2, \dots, \infty\}$ is a countable compact set endowed with the following metric: $d(k, m) = |\frac{1}{k} - \frac{1}{m}|$ for any $1 \leq k, m \leq \infty$. The sequence $\{Q_{t+1}^\tau\}_{1 \leq \tau \leq \infty}$ can be regarded as a correspondence Q_{t+1} from $H_t \times \{1, 2, \dots, \infty\}$ to \mathbb{R}^n , which is measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \{1, 2, \dots, \infty\}$. The backward induction in Subsection 5.3.1 shows that $\Phi(Q_{t+1})$ is measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \{1, 2, \dots, \infty\}$.

Since $\Phi(Q_{t+1})$ is essentially sectionally upper hemicontinuous on $X^t \times \{1, 2, \dots, \infty\}$, there exists a measurable subset $\check{S}^{t-1} \subseteq S^{t-1}$ such that $\lambda^{t-1}(\check{S}^{t-1}) = 1$, and $\Phi(Q_{t+1})(\cdot, \cdot, \check{s}^{t-1})$ is upper hemicontinuous for any $\check{s}^{t-1} \in \check{S}^{t-1}$. Fix $\check{s}^{t-1} \in \check{S}^{t-1}$. For $h_{t-1} = (x^{t-1}, \check{s}^{t-1}) \in H_{t-1}$ and $a \in Q_t^\infty(h_{t-1})$, by its definition, $a \in Q_t^\tau(h_{t-1}) = \Phi(Q_{t+1}^\tau)(h_{t-1})$ for $\tau \geq t$. Thus, $a \in \Phi(Q_{t+1}^\infty)(h_{t-1})$.

In summary, $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1})$ for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$. \square

Though the definition of Q_t^τ involves correlated strategies for $\tau < t$, the following lemma shows that one can work with mixed strategies in terms of equilibrium payoffs via the combination of backward and forward inductions in multiple steps.

Lemma 21. *If c_t is a measurable selection of $\Phi(Q_{t+1}^\infty)$, then $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$.*

Proof. Without loss of generality, we only prove the case $t = 1$.

Suppose that c_1 is a measurable selection of $\Phi(Q_2^\infty)$. Apply Proposition 5 recursively to obtain Borel measurable mappings $\{f_{ki}\}_{i \in I}$ for $k \geq 1$. That is, for any $k \geq 1$, there exists a Borel measurable selection c_k of Q_k^∞ such that for λ_{k-1} -almost all $h_{k-1} \in H_{k-1}$,

1. $f_k(h_{k-1})$ is a Nash equilibrium in the subgame h_{k-1} , where the action space

is $A_{ki}(h_{k-1})$ for player $i \in I$, and the payoff function is given by

$$\int_{S_k} c_{k+1}(h_{k-1}, \cdot, s_k) f_{k0}(ds_k|h_{k-1}).$$

2.

$$c_k(h_{k-1}) = \int_{A_k(h_{k-1})} \int_{S_k} c_{k+1}(h_{k-1}, x_k, s_k) f_{k0}(ds_k|h_{k-1}) f_k(dx_k|h_{k-1}).$$

We need to show that $c_1(h_0)$ is a subgame-perfect equilibrium payoff vector for λ_0 -almost all $h_0 \in H_0$.

Step 1. We show that for any $k \geq 1$ and λ_{k-1} -almost all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{k-1}) = \int_{\prod_{m \geq k} (X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(d(x, s)).$$

Since the game is continuous at infinity, there exists some positive integer $M > k$ such that w^M is sufficiently small. By Lemma 20, $c_k(h_{k-1}) \in Q_k^\infty(h_{k-1}) = \cap_{\tau \geq 1} Q_k^\tau(h_{k-1})$ for λ_{k-1} -almost all $h_{k-1} \in H_{k-1}$. Since $Q_k^\tau = \Phi^{\tau-k+1}(Q_{\tau+1}^\tau)$ for $k \leq \tau$, $c_k(h_{k-1}) \in \cap_{\tau \geq k} \Phi^{\tau-k+1}(Q_{\tau+1}^\tau)(h_{k-1}) \subseteq \Phi^{M-k+1}(Q_{M+1}^M)(h_{k-1})$ for λ_{k-1} -almost all $h_{k-1} \in H_{k-1}$. Thus, there exists a Borel measurable selection w of Q_{M+1}^M and some $\xi \in \Upsilon$ such that for λ_{M-1} -almost all $h_{M-1} \in H_{M-1}$,

i. $f_M(h_{M-1})$ is a Nash equilibrium in the subgame h_{M-1} , where the action space is $A_{Mi}(h_{M-1})$ for player $i \in I$, and the payoff function is given by

$$\int_{S_M} w(h_{M-1}, \cdot, s_M) f_{M0}(ds_M|h_{M-1});$$

ii.

$$c_M(h_{M-1}) = \int_{A_M(h_{M-1})} \int_{S_M} w(h_{M-1}, x_M, s_M) f_{M0}(ds_M|h_{M-1}) f_M(dx_M|h_{M-1});$$

iii. $w(h_M) = \int_{\prod_{m \geq M+1} (X_m \times S_m)} u(h_M, x, s) \varrho_{(h_M, \xi)}(d(x, s))$.

Then for λ_{k-1} -almost all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{k-1}) = \int_{\prod_{m \geq k} (X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f^M)}(d(x, s)),$$

where f_k^M is f_k if $k \leq M$, and ξ_k if $k \geq M + 1$. Since the game is continuous at

infinity,

$$\int_{\prod_{m \geq k} (X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f^M)}(\mathbf{d}(x, s))$$

converges to

$$\int_{\prod_{m \geq k} (X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathbf{d}(x, s))$$

when M goes to infinity. Thus, for λ_{k-1} -almost all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{k-1}) = \int_{\prod_{m \geq k} (X_m \times S_m)} u(h_{k-1}, x, s) \varrho_{(h_{k-1}, f)}(\mathbf{d}(x, s)). \quad (4)$$

Step 2. Below, we show that $\{f_{ki}\}_{i \in I}$ is a subgame-perfect equilibrium.

Fix a player i and a strategy $g_i = \{g_{ki}\}_{k \geq 1}$. For each $k \geq 1$, define a new strategy \tilde{f}_i^k as follows: $\tilde{f}_i^k = (g_{1i}, \dots, g_{ki}, f_{(k+1)i}, f_{(k+2)i}, \dots)$. That is, we simply replace the initial k stages of f_i by g_i . Denote $\tilde{f}^k = (\tilde{f}_i^k, f_{-i})$.

Fix $k \geq 1$ and a measurable subset $D^k \subseteq S^k$ such that (1) and (2) of step 1 and Equation (4) hold for all $s_k \in D^k$ and $x^k \in H_k(s^k)$, and $\lambda^k(D^k) = 1$. For each $\tilde{M} > k$, by the Fubini property, there exists a measurable subset $E_k^{\tilde{M}} \subseteq S^k$ such that $\lambda^k(E_k^{\tilde{M}}) = 1$ and $\otimes_{k+1 \leq j \leq \tilde{M}} \lambda_j(D^{\tilde{M}}(s^k)) = 1$ for all $s^k \in E_k^{\tilde{M}}$, where

$$D^{\tilde{M}}(s^k) = \{(s_{k+1}, \dots, s_{\tilde{M}}) : (s^k, s_{k+1}, \dots, s_{\tilde{M}}) \in D^{\tilde{M}}\}.$$

Let $\hat{D}^k = (\cap_{\tilde{M} > k} E_k^{\tilde{M}}) \cap D^k$. Then $\lambda^k(\hat{D}^k) = 1$.

For any $h_k = (x^k, s^k)$ such that $s^k \in \hat{D}^k$ and $x^k \in H_k(s^k)$, we have

$$\begin{aligned} & \int_{\prod_{m \geq k+1} (X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, f)}(\mathbf{d}(x, s)) \\ &= \int_{A_{k+1}(h_k)} \int_{S_{k+1}} c_{(k+2)i}(h_k, x_{k+1}, s_{k+1}) f_{(k+1)0}(\mathbf{d}s_{k+1}|h_k) f_{k+1}(\mathbf{d}x_{k+1}|h_k) \\ &\geq \int_{A_{k+1}(h_k)} \int_{S_{k+1}} c_{(k+2)i}(h_k, x_{k+1}, s_{k+1}) f_{(k+1)0}(\mathbf{d}s_{k+1}|h_k) (f_{(k+1)(-i)} \otimes g_{(k+1)i})(\mathbf{d}x_{k+1}|h_k) \\ &= \int_{A_{k+1}(h_k)} \int_{S_{k+1}} \int_{A_{k+2}(h_k, x_{k+1}, s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\ &\quad f_{(k+2)0}(\mathbf{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes f_{(k+2)i}(\mathbf{d}x_{k+2}|h_k, x_{k+1}, s_{k+1}) \\ &\quad f_{(k+1)0}(\mathbf{d}s_{k+1}|h_k) f_{(k+1)(-i)} \otimes g_{(k+1)i}(\mathbf{d}x_{k+1}|h_k) \\ &\geq \int_{A_{k+1}(h_k)} \int_{S_{k+1}} \int_{A_{k+2}(h_k, x_{k+1}, s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\ &\quad f_{(k+2)0}(\mathbf{d}s_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes g_{(k+2)i}(\mathbf{d}x_{k+2}|h_k, x_{k+1}, s_{k+1}) \\ &\quad f_{(k+1)0}(\mathbf{d}s_{k+1}|h_k) f_{(k+1)(-i)} \otimes g_{(k+1)i}(\mathbf{d}x_{k+1}|h_k) \end{aligned}$$

$$= \int_{\prod_{m \geq k+1} (X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, \tilde{f}^{k+2})}(\mathbf{d}(x, s)).$$

The first and the last equalities follow from Equation (4) in the end of step 1. The second equality is due to (2) in step 1. The first inequality is based on (1) in step 1. The second inequality holds by the following arguments:

- i. by the choice of h_k and (1) in step 1, for λ_{k+1} -almost all $s_{k+1} \in S_{k+1}$ and all $x_{k+1} \in X_{k+1}$ such that $(h_k, x_{k+1}, s_{k+1}) \in H_{k+1}$, we have

$$\begin{aligned} & \int_{A_{k+2}(h_k, x_{k+1}, s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\ & f_{(k+2)0}(\mathrm{d}s_{k+2} | h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes f_{(k+2)i}(\mathrm{d}x_{k+2} | h_k, x_{k+1}, s_{k+1}) \\ & \geq \int_{A_{k+2}(h_k, x_{k+1}, s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\ & f_{(k+2)0}(\mathrm{d}s_{k+2} | h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes g_{(k+2)i}(\mathrm{d}x_{k+2} | h_k, x_{k+1}, s_{k+1}); \end{aligned}$$

- ii. since $f_{(k+1)0}$ is absolutely continuous with respect to λ_{k+1} , the above inequality also holds for $f_{(k+1)0}(h_k)$ -almost all $s_{k+1} \in S_{k+1}$ and all $x_{k+1} \in X_{k+1}$ such that $(h_k, x_{k+1}, s_{k+1}) \in H_{k+1}$.

Repeating the above argument, one can show that

$$\begin{aligned} & \int_{\prod_{m \geq k+1} (X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, f)}(\mathbf{d}(x, s)) \\ & \geq \int_{\prod_{m \geq k+1} (X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, \tilde{f}^{\tilde{M}+1})}(\mathbf{d}(x, s)) \end{aligned}$$

for any $\tilde{M} > k$. Since

$$\int_{\prod_{m \geq k+1} (X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, \tilde{f}^{\tilde{M}+1})}(\mathbf{d}(x, s))$$

converges to

$$\int_{\prod_{m \geq k+1} (X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, (g_i, f_{-i}))}(\mathbf{d}(x, s))$$

as \tilde{M} goes to infinity, we have

$$\begin{aligned} & \int_{\prod_{m \geq k+1} (X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, f)}(\mathbf{d}(x, s)) \\ & \geq \int_{\prod_{m \geq k+1} (X_m \times S_m)} u(h_k, x, s) \varrho_{(h_k, (g_i, f_{-i}))}(\mathbf{d}(x, s)). \end{aligned}$$

Therefore, $\{f_{ki}\}_{i \in I}$ is a subgame-perfect equilibrium. \square

By Lemma 20 and Proposition 4, the correspondence $\Phi(Q_{t+1}^\infty)$ is measurable, nonempty and compact valued. By Lemma 3 (3), it has a measurable selection. Then Theorem 1 follows from the above lemma.

For $t \geq 1$ and $h_{t-1} \in H_{t-1}$, recall that $E_t(h_{t-1})$ is the set of payoff vectors of subgame-perfect equilibria in the subgame h_{t-1} . The following lemma shows that $E_t(h_{t-1})$ is essentially the same as $Q_t^\infty(h_{t-1})$.

Lemma 22. *For any $t \geq 1$, $E_t(h_{t-1}) = Q_t^\infty(h_{t-1})$ for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$.*

Proof. (1) We will first prove the following claim: for any t and τ , if $E_{t+1}(h_t) \subseteq Q_{t+1}^\tau(h_t)$ for λ^t -almost all $h_t \in H_t$, then $E_t(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$ for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$. We only need to consider the case that $t \leq \tau$.

By the construction of $\Phi(Q_{t+1}^\tau)$ in Subsection 5.3.1, there exists a measurable subset $\acute{S}^{t-1} \subseteq S^{t-1}$ with $\lambda^{t-1}(\acute{S}^{t-1}) = 1$ such that for any c_t and $h_{t-1} = (x^{t-1}, s^{t-1}) \in H_{t-1}$ with $s^{t-1} \in \acute{S}^{t-1}$, if

1. $c_t = \int_{A_t(h_{t-1})} \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}) \alpha(dx_t)$, where $q_{t+1}(h_{t-1}, \cdot)$ is measurable and $q_{t+1}(h_{t-1}, x_t, s_t) \in Q_{t+1}^\tau(h_{t-1}, x_t, s_t)$ for λ_t -almost all $s_t \in S_t$ and $x_t \in A_t(h_{t-1})$;
2. $\alpha \in \otimes_{i \in I} \mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame h_{t-1} with payoff $\int_{S_t} q_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(ds_t|h_{t-1})$ and action space $\prod_{i \in I} A_{ti}(h_{t-1})$,

then $c_t \in \Phi(Q_{t+1}^\tau)(h_{t-1})$.

Fix a subgame $h_{t-1} = (x^{t-1}, s^{t-1})$ such that $s^{t-1} \in \acute{S}^{t-1}$. Pick a point $c_t \in E_t(\acute{s}^{t-1})$. There exists a strategy profile f such that f is a subgame-perfect equilibrium in the subgame h_{t-1} and the payoff is c_t . Let $c_{t+1}(h_{t-1}, x_t, s_t)$ be the payoff vector induced by $\{f_{ti}\}_{i \in I}$ in the subgame $(h_t, x_t, s_t) \in \text{Gr}(A_t) \times S_t$. Then we have

1. $c_t = \int_{A_t(h_{t-1})} \int_{S_t} c_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}) f_t(dx_t|h_{t-1})$;
2. $f_t(\cdot|h_{t-1})$ is a Nash equilibrium in the subgame h_{t-1} with action space $A_t(h_{t-1})$ and payoff $\int_{S_t} c_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(ds_t|h_{t-1})$.

Since f is a subgame-perfect equilibrium in the subgame h_{t-1} , $c_{t+1}(h_{t-1}, x_t, s_t) \in E_{t+1}(h_{t-1}, x_t, s_t) \subseteq Q_{t+1}^\tau(h_{t-1}, x_t, s_t)$ for λ_t -almost all $s_t \in S_t$ and $x_t \in A_t(h_{t-1})$, which implies that $c_t \in \Phi(Q_{t+1}^\tau)(h_{t-1}) = Q_t^\tau(h_{t-1})$.

Therefore, $E_t(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$ for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$.

(2) For any $t > \tau$, $E_t \subseteq Q_t^\tau$. If $t \leq \tau$, we can start with $E_{\tau+1} \subseteq Q_{\tau+1}^\tau$ and repeat the argument in (1), then we can show that $E_t(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$ for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$. Thus, $E_t(h_{t-1}) \subseteq Q_t^\infty(h_{t-1})$ for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$.

(3) Suppose that c_t is a measurable selection from $\Phi(Q_{t+1}^\infty)$. Apply Proposition 5 recursively to obtain Borel measurable mappings $\{f_{ki}\}_{i \in I}$ for $k \geq t$. By

Lemma 21, $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$. Consequently, $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq E_t(h_{t-1})$ for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$.

By Lemma 20, $E_t(h_{t-1}) = Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1})$ for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$. \square

Therefore, we have proved Proposition 1.

5.4 Proof of Proposition 2

We will highlight the needed changes in comparison with the proofs presented in Subsections 5.3.1-5.3.3.

1. Backward induction. We first consider stage t with $N_t = 1$.

If $N_t = 1$, then $S_t = \{s_t\}$. Thus, $P_t(h_{t-1}, x_t) = Q_{t+1}(h_{t-1}, x_t, s_t)$, which is nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$. Notice that P_t may not be convex valued.

We first assume that P_t is upper hemicontinuous. Suppose that j is the player who is active in this period. Consider the correspondence $\Phi_t: H_{t-1} \rightarrow \mathbb{R}^n \times \mathcal{M}(X_t) \times \Delta(X_t)$ defined as follows: $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x_{tj}^*)$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$,³⁵
2. $x_{tj}^* \in A_{tj}(h_{t-1})$ is a maximization point of player j given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x_{tj}^*}$;
3. $\mu = \delta_{p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x_{tj}^*)}$.

This is a single agent problem. We need to show that Φ_t is nonempty and compact valued, and upper hemicontinuous.

If P_t is nonempty, convex and compact valued, and upper hemicontinuous, then we can use Lemma 16, the main result of Simon and Zame (1990), to prove the nonemptiness, compactness, and upper hemicontinuity of Φ_t . In Simon and Zame (1990), the only step they need the convexity of P_t for the proof of their main theorem is Lemma 2 therein. However, the one-player pure-strategy version of their Lemma 2, stated in the following, directly follows from the upper hemicontinuity of P_t without requiring the convexity.

Let Z be a compact metric space, and $\{z_n\}_{n \geq 0} \subseteq Z$. Let $P: Z \rightarrow \mathbb{R}_+$ be a bounded, upper hemicontinuous correspondence with nonempty and compact values. For each $n \geq 1$, let q_n be a Borel measurable selection

³⁵Note that $A_{t(-j)}$ is point valued since all players other than j are inactive.

of P such that $q_n(z_n) = d_n$. If z_n converges to z_0 and d_n converges to some d_0 , then $d_0 \in P(z_0)$.

Repeat the argument in the proof of the main theorem of [Simon and Zame \(1990\)](#), one can show that Φ_t is nonempty and compact valued, and upper hemicontinuous.

Then we go back to the case that P_t is nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$. Recall that we proved Proposition 4 based on Lemma 16. If P_t is essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$, we can show the following result based on a similar argument as in Subsections 5.2: there exists a bounded, measurable, nonempty and compact valued correspondence Φ_t from H_{t-1} to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \Delta(X_t)$ such that Φ_t is essentially sectionally upper hemicontinuous on $X^{t-1} \times \hat{S}^{t-1}$, and for λ^{t-1} -almost all $h_{t-1} \in H_{t-1}$, $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x_{tj}^*)$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$;
2. $x_{tj}^* \in A_{tj}(h_{t-1})$ is a maximization point of player j given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x_{tj}^*}$;
3. $\mu = \delta_{p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x_{tj}^*)}$.

Next we consider the case that $N_t = 0$. Suppose that the correspondence Q_{t+1} from H_t to \mathbb{R}^n is bounded, measurable, nonempty and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^t$. For any $(h_{t-1}, x_t, \hat{s}_t) \in \text{Gr}(\hat{A}_t)$, let

$$\begin{aligned} R_t(h_{t-1}, x_t, \hat{s}_t) &= \int_{\tilde{S}_t} Q_{t+1}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t) \tilde{f}_{t0}(d\tilde{s}_t | h_{t-1}, x_t, \hat{s}_t) \\ &= \int_{\tilde{S}_t} Q_{t+1}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t) \varphi_{t0}(h_{t-1}, x_t, \hat{s}_t, \tilde{s}_t) \lambda_t(d\tilde{s}_t). \end{aligned}$$

Then following the same argument as in Subsection 5.3.1, one can show that R_t is a nonempty, convex and compact valued, and essentially sectionally upper hemicontinuous correspondence on $X^t \times \hat{S}^t$.

For any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$, let

$$P_t(h_{t-1}, x_t) = \int_{\hat{A}_{t0}(h_{t-1}, x_t)} R_t(h_{t-1}, x_t, \hat{s}_t) \hat{f}_{t0}(d\hat{s}_t | h_{t-1}, x_t).$$

By Lemma 8, P_t is nonempty, convex and compact valued, and essentially sectionally upper hemicontinuous on $X^t \times \hat{S}^{t-1}$. The rest of the step remains the same as in Subsection 5.3.1.

2. Forward induction: unchanged.

3. Infinite horizon: we need to slightly modify the definition of $\Xi_t^{m_1}$ for any $m_1 \geq t \geq 1$. Fix any $t \geq 1$. Define a correspondence Ξ_t^t as follows: in the subgame h_{t-1} ,

$$\Xi_t^t(h_{t-1}) = (\mathcal{M}(A_t(h_{t-1})) \diamond \hat{f}_{t0}(h_{t-1}, \cdot)) \otimes \lambda_t.$$

For any $m_1 > t$, suppose that the correspondence $\Xi_t^{m_1-1}$ has been defined. Then we can define a correspondence $\Xi_t^{m_1} : H_{t-1} \rightarrow \mathcal{M}\left(\prod_{t \leq m \leq m_1} (X_m \times S_m)\right)$ as follows:

$$\begin{aligned} \Xi_t^{m_1}(h_{t-1}) = & \{g(h_{t-1}) \diamond ((\xi_{m_1}(h_{t-1}, \cdot) \diamond \hat{f}_{m_1 0}(h_{t-1}, \cdot)) \otimes \lambda_{m_1}) : \\ & g \text{ is a Borel measurable selection of } \Xi_t^{m_1-1}, \\ & \xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1})\}. \end{aligned}$$

Then the result in Subsection 5.3.3 is true with the above $\Xi_t^{m_1}$.

Consequently, a subgame-perfect equilibrium exists.

5.5 Proof of Proposition 3

We will describe the necessary changes in comparison with the proofs presented in Subsections 5.3.1-5.3.3 and 5.4.

1. Backward induction. For any $t \geq 1$, suppose that the correspondence Q_{t+1} from H_t to \mathbb{R}^n is bounded, nonempty and compact valued, and upper hemicontinuous on X^t .

If $N_t = 1$, then $S_t = \{\acute{s}_t\}$. Thus, $P_t(h_{t-1}, x_t) = Q_{t+1}(h_{t-1}, x_t, \acute{s}_t)$, which is nonempty and compact valued, and upper hemicontinuous. Then define the correspondence Φ_t from H_{t-1} to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \Delta(X_t)$ as $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if

1. $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x_{tj}^*)$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$;
2. $x_{tj}^* \in A_{tj}(h_{t-1})$ is a maximization point of player j given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x_{tj}^*}$;
3. $\mu = \delta_{p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x_{tj}^*)}$.

As discussed in Subsection 5.4, Φ_t is nonempty and compact valued, and upper hemicontinuous.

When $N_t = 0$, for any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$,

$$P_t(h_{t-1}, x_t) = \int_{A_{t0}(h_{t-1}, x_t)} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t | h_{t-1}, x_t).$$

Let $\text{co}Q_{t+1}(h_{t-1}, x_t, s_t)$ be the convex hull of $Q_{t+1}(h_{t-1}, x_t, s_t)$. Because Q_{t+1} is bounded, nonempty and compact valued, $\text{co}Q_{t+1}$ is bounded, nonempty, convex and compact valued. By Lemma 3 (8), $\text{co}Q_{t+1}$ is upper hemicontinuous.

Notice that $f_{t0}(\cdot | h_{t-1}, x_t)$ is atomless and Q_{t+1} is nonempty and compact valued. By Lemma 5,

$$P_t(h_{t-1}, x_t) = \int_{A_{t0}(h_{t-1}, x_t)} \text{co}Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t | h_{t-1}, x_t).$$

By Lemma 8, P_t is bounded, nonempty, convex and compact valued, and upper hemicontinuous. Then instead of relying on Proposition 4, we now use Lemma 16 to conclude that Φ_t is bounded, nonempty and compact valued, and upper hemicontinuous.

2. Forward induction. The first step is much simpler.

For any $\{(h_{t-1}^k, v^k)\}_{1 \leq k \leq \infty} \subseteq \text{Gr}(\Phi_t(Q_{t+1}))$ such that (h_{t-1}^k, v^k) converges to $(h_{t-1}^\infty, v^\infty)$, pick (α^k, μ^k) such that $(v^k, \alpha^k, \mu^k) \in \Phi_t(h_{t-1}^k)$ for $1 \leq k < \infty$. Since Φ_t is upper hemicontinuous and compact valued, there exists a subsequence of (v^k, α^k, μ^k) , say itself, such that (v^k, α^k, μ^k) converges to some $(v^\infty, \alpha^\infty, \mu^\infty) \in \Phi_t(h_{t-1}^\infty)$ due to Lemma 2 (5). Thus, $(\alpha^\infty, \mu^\infty) \in \Psi_t(h_{t-1}^\infty, v^\infty)$, which implies that Ψ_t is also upper hemicontinuous and compact valued. By Lemma 3 (3), Ψ_t has a Borel measurable selection ψ_t . Given a Borel measurable selection q_t of $\Phi(Q_{t+1})$, one can let $\phi_t(h_{t-1}) = (q_t(h_{t-1}), \psi_t(h_{t-1}, q_t(h_{t-1})))$. Then ϕ_t is a Borel measurable selection of Φ_t .

Steps 2 and 3 are unchanged.

3. Infinite horizon. We do not need to consider $\Xi_t^{m_1}$ for any $m_1 \geq t \geq 1$. Instead of relying on Lemma 14, we can use Lemma 10 (3) to prove Lemma 18. The proof of Lemma 19 is much simpler. Notice that the boundedness, nonemptiness, compactness and upper hemicontinuity of Q_t^τ for the case $t > \tau$ is immediate. Then one can apply the backward induction as in Lemma 19 to show the corresponding properties of Q_t^τ for the case $t \leq \tau$. Following the same arguments, one can show that Lemmas 20-22 now hold for all $h_{t-1} \in H_{t-1}$ and all $t \geq 1$.

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